

SPECTRAL ANALYSIS OF SUBORDINATE BROWNIAN MOTIONS IN HALF-LINE

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ABSTRACT. We study one-dimensional Lévy processes with Lévy-Khintchine exponent $\psi(\xi^2)$, where ψ is a complete Bernstein function. These processes are subordinate Brownian motions corresponding to subordinators, whose Lévy measure has completely monotone density; or, equivalently, symmetric Lévy processes whose Lévy measure has completely monotone density on $(0, \infty)$. Examples include symmetric stable processes and relativistic processes. The main result is a formula for the generalized eigenfunctions of transition operators of the process killed after exiting the half-line. A generalized eigenfunction expansion of the transition operators is derived. As an application, a formula for the distribution of the first passage time (or the supremum functional) is obtained.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In a recent paper [49], spectral problem for the one-dimensional Cauchy process (that is, the symmetric 1-stable process) killed upon exiting a half-line or an interval was studied. For the half-line, an explicit formula for generalized eigenfunctions of transition operators was obtained using methods developed in the theory of linear water waves. The argument of [49] relies on some properties specific to the Cauchy process, and it does not easily generalize to other Lévy processes. The purpose of this article is to derive a similar formula for generalized eigenfunctions in a more general setting, using a modified method.

The class of processes considered here consists of symmetric (one-dimensional) Lévy processes X_t with Lévy measure having completely monotone density function on $(0, \infty)$. Alternatively, this class can be described as (one-dimensional) subordinate Brownian motions, corresponding to subordinators with Lévy measure having completely monotone density function. Yet another characterization is given by the condition that the Lévy-Khintchine exponent of X_t has the form $\psi(\xi^2)$ for a *complete Bernstein function* $\psi(\xi)$. The equivalence of the above is given in Proposition 2.14. This class of Lévy processes have attracted much attention in the last decade;

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see, for example, [10, 16, 17, 35, 40, 41, 43, 44, 68, 69] for some recent developments. There is also an extensive literature focused specifically on symmetric α -stable processes ([2, 3, 9, 11, 15, 19, 20, 21, 36, 47, 48, 59]) and relativistic α -stable processes ([12, 13, 14, 50, 63]), which are included here as examples.

We consider the process X_t killed upon leaving the half-line $(0, \infty)$. Let $P_t^{(0, \infty)}$, $\mathcal{A}_{(0, \infty)}$ and $\mathcal{D}(\mathcal{A}_{(0, \infty)}; L^\infty)$ denote the corresponding transition operators, the $L^\infty((0, \infty))$ generator, and the domain of $\mathcal{A}_{(0, \infty)}$, respectively (formal definitions are given in the Preliminaries). The following generalization of Theorem 2 from [49] is the main result of the article. Its main advantage is the explicit description of the eigenfunctions.

Theorem 1.1. *Suppose that the Lévy-Khintchine exponent of X_t has the form $\psi(\xi^2)$ for a complete Bernstein function ψ . For all $\lambda > 0$, there is a bounded function F_λ on $(0, \infty)$ which is the eigenfunction of $P_t^{(0, \infty)}$ and $\mathcal{A}_{(0, \infty)}$:*

$$P_t^{(0, \infty)} F_\lambda(x) = e^{-t\psi(\lambda^2)} F_\lambda(x) \quad \text{and} \quad \mathcal{A}_{(0, \infty)} F_\lambda(x) = -\psi(\lambda^2) F_\lambda(x)$$

for all $x > 0$. The function F_λ is characterized by its Laplace transform:

$$(1.1) \quad \mathcal{L}F_\lambda(\xi) = \frac{\lambda}{\lambda^2 + \xi^2} \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + \zeta^2} \log \frac{\psi'(\lambda^2)(\lambda^2 - \zeta^2)}{\psi(\lambda^2) - \psi(\zeta^2)} d\zeta\right)$$

for $\xi \in \mathbf{C}$ such that $\operatorname{Re} \xi > 0$. Furthermore, for $x > 0$ we have

$$(1.2) \quad F_\lambda(x) = \sin(\lambda x + \vartheta_\lambda) - G_\lambda(x),$$

where $\vartheta_\lambda \in [0, \pi/2)$, and $G_\lambda(x)$ is a bounded, completely monotone function on $(0, \infty)$. More precisely, we have

$$(1.3) \quad \vartheta_\lambda = -\frac{1}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 - \zeta^2} \log \frac{\psi'(\lambda^2)(\lambda^2 - \zeta^2)}{\psi(\lambda^2) - \psi(\zeta^2)} d\zeta,$$

and G_λ is the Laplace transform of a finite measure γ_λ on $(0, \infty)$. When $\psi(\xi)$ extends to a function $\psi^+(\xi)$ holomorphic in the upper complex half-plane $\{\xi \in \mathbf{C} : \operatorname{Im} \xi > 0\}$ and continuous in $\{\xi \in \mathbf{C} : \operatorname{Im} \xi \geq 0\}$, and furthermore $\psi^+(-\xi) \neq \psi(\lambda)$ for all $\xi > 0$, then the measure γ_λ is absolutely continuous, and

$$(1.4) \quad \begin{aligned} \gamma_\lambda(d\xi) = & \frac{1}{\pi} \left(\operatorname{Im} \frac{\lambda \psi'(\lambda^2)}{\psi(\lambda^2) - \psi^+(-\xi^2)} \right) \\ & \times \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + \zeta^2} \log \frac{\psi'(\lambda^2)(\lambda^2 - \zeta^2)}{\psi(\lambda^2) - \psi(\zeta^2)} d\zeta\right) d\xi \end{aligned}$$

for $\xi > 0$.

In some cases, the above formulae can be substantially simplified. For example, when X_t is the symmetric α -stable process, then $\vartheta_\lambda = (2 - \alpha)\pi/8$, and $F_\lambda(x) = F_1(\lambda x)$; see Section 6.

Any complete Bernstein function $\psi(\xi)$ extends holomorphically to $\mathbf{C} \setminus (-\infty, 0]$. Hence, the assumption in the last part of Theorem 1.1 concerns the existence of boundary values of ψ (approached from the upper complex

half-plane) along the branch cut $(-\infty, 0]$. A formula similar to (1.4) can be given in the general case, as discussed in Remark 4.13.

It should be emphasized that although Theorem 1.1 extends the result of [49], its proof is essentially different.

Remark 1.2. Entirely analytical formulation of Theorem 1.1 is available; see [37] for the analytical definition of $\mathcal{A}_{(0,\infty)}$. Also, our arguments are purely analytical, with the exception of probabilistic proofs of Lemmas 2.10 and 2.11. \square

The eigenfunctions F_λ *never* belong to $L^2((0, \infty))$: this reflects the fact that the spectrum of transition operators $P_t^{(0,\infty)}$, considered as operators on $L^2((0, \infty))$, is purely continuous. Nevertheless, the functions F_λ yield a *generalized eigenfunction expansion* of $P_t^{(0,\infty)}$. In classical eigenfunction expansion, a function is decomposed with respect to a complete orthonormal set of eigenfunctions, and the operator acts at each component independently. Informally, in *generalized* eigenfunction expansions, the principle is the same, but eigenfunctions are no longer square-integrable and there are uncountably many of them. A formal statement is given in the following result, which generalizes Theorem 3 in [49], and explicitly provides functional calculus for $\mathcal{A}_{(0,\infty)}$. Here $\mathcal{A}_{(0,\infty)}$ and $\mathcal{D}(\mathcal{A}_{(0,\infty)}; L^2)$ denote the $L^2((0, \infty))$ generator of the semigroup $P_t^{(0,\infty)}$, and the domain of $\mathcal{A}_{(0,\infty)}$, respectively (again, see Preliminaries for formal definitions).

Theorem 1.3. *With the notation of Theorem 1.1, let*

$$(1.5) \quad \Pi f(\lambda) = \int_0^\infty f(x) F_\lambda(x) dx \quad \text{for } \lambda > 0, f \in C_c((0, \infty)).$$

Then $\sqrt{2/\pi} \Pi$ extends to a unitary operator on $L^2((0, \infty))$, and

$$\Pi P_t^{(0,\infty)} f(\lambda) = e^{-t\psi(\lambda^2)} \Pi f(\lambda) \quad \text{for } f \in L^2((0, \infty)).$$

Furthermore, $f \in \mathcal{D}(\mathcal{A}_{(0,\infty)}; L^2)$ if and only if $\psi(\lambda^2) \Pi f(\lambda)$ is in $L^2((0, \infty))$, and

$$\Pi \mathcal{A}_{(0,\infty)} f(\lambda) = -\psi(\lambda^2) \Pi f(\lambda) \quad \text{for } f \in \mathcal{D}(\mathcal{A}_{(0,\infty)}; L^2).$$

Remark 1.4. Below we provide a complete proof of Theorem 1.3 under an additional condition that Π is injective, and we verify that Π is injective when X_t is the symmetric α -stable process, $\alpha \in (0, 2]$ (that is, $\psi(\xi) = \xi^{\alpha/2}$). In the preliminary version of this article, a relatively easy to check, but rather restrictive sufficient condition for injectivity of Π was given (Lemma 19 in [53]), and it was conjectured that in fact Π is always injective. After a few months, this conjecture was solved in affirmative by Jacek Małecki, and the proof of injectivity of Π is given in the recent preprint [56]. We announce the result of [56] and state Theorem 1.3 in full generality. \square

Remark 1.5. There is a striking similarity between formulae (1.1) and (1.4) on one hand, and some formulae in fluctuation theory of Lévy processes on the other. More precisely, the exponent in (1.1) and (1.4) resembles known

formulae for the Laplace exponent of ladder processes (see Corollary 9.7 in [28]) and the Laplace transform of the supremum functional (Theorem 1 in [4]). This phenomenon can be easily explained at the analytical level: in both cases explicit formulae are obtained using the *Wiener-Hopf method*, a technique for solving some integral equations in the half-line via Fourier transform. Fluctuation theory of Lévy processes relies upon the Wiener-Hopf factorization of the λ -potential operator (that is, the resolvent) of the transition semigroup of a Lévy process. The Fourier symbol of this operator is $1/(\lambda + \psi(\xi^2))$. In the proof of Theorem 1.1, the operator with Fourier symbol $(\psi(\lambda^2) - \psi(\xi^2))/(\lambda^2 - \xi^2)$ is factorized.

A probabilistic formulation of the fluctuation theory is available through local times, excursion theory and ladder processes; see [5, 23, 57, 64]. It is an interesting open problem whether there is a similar *probabilistic* derivation of the formula for F_λ . \square

The proof of Theorem 1.3 (under the assumption that Π is injective) follows the line of [49]. Proofs of Theorems 1.1 and 1.3 are rather technical. Therefore, it may be helpful to keep in mind the following well-recognized example.

Example 1.6. Suppose that X_t is the Brownian motion, with variance $2t$ (so that the generator of X_t is the one-dimensional Laplace operator d^2/dx^2 , and $\psi(\xi) = \xi$). In this case $\mathcal{A}_{(0,\infty)}$ is the Laplace operator in $(0, \infty)$ with Dirichlet boundary condition at 0, and $P_t^{(0,\infty)}$ is the classical heat semigroup on $(0, \infty)$ with Dirichlet boundary condition at 0. The eigenfunctions of $\mathcal{A}_{(0,\infty)}$ and $P_t^{(0,\infty)}$ are simply $F_\lambda(x) = \sin(\lambda x)$, with corresponding eigenvalues $-\lambda^2$ and $e^{-t\lambda^2}$, respectively. The integral transform Π is the Fourier sine transform on $(0, \infty)$, and Theorem 1.3 states that the Fourier sine transform is the unitary mapping (up to a constant factor $\sqrt{2/\pi}$), which diagonalizes the action of the heat semigroup.

To the author's knowledge, no results similar to Theorems 1.1 and 1.3 were available for Lévy processes other than the Brownian motion (possibly with drift; this requires a minor modification to the above example) and the Cauchy process (studied in [49]). Although generalized eigenfunction expansions similar to Theorem 1.3 have been studied for various classes of operators (see [30] for the case of general Markov processes, and [61, 67] for diffusion processes with Feynman-Kac potential), the explicit description of eigenfunctions was not available. Consequently, their applications were limited. The novelty of Theorem 1.3 is in that the generalized eigenfunctions $F_\lambda(x)$ are given in a fairly explicit form, allowing various estimates, asymptotic analysis and numerical approximation. For this reason, one can expect that Theorems 1.1 and 1.3 will find a variety of applications; some are already discussed below.

Theorems 1.1 and 1.3 yield formulae for the transition density $p_t^{(0,\infty)}(x, y)$ of X_t killed upon leaving the half-line $(0, \infty)$, and for the distribution of the first exit time $\tau_{(0,\infty)}$ from the positive half-line. These results are contained

in the following two theorems. By \mathbf{P}_x we denote the probability for the process X_t starting at a fixed point $x > 0$.

Theorem 1.7. *Let $t > 0$. If $e^{-t\psi(\xi^2)}$ is integrable in $\xi > 0$, then*

$$(1.6) \quad p_t^{(0,\infty)}(x, y) = \frac{2}{\pi} \int_0^\infty e^{-t\psi(\lambda^2)} F_\lambda(x) F_\lambda(y) d\lambda \quad \text{for } x, y > 0.$$

Theorem 1.8. *Let $t > 0$. If*

$$(1.7) \quad \sup_{\xi > 0} \frac{\xi |\psi''(\xi)|}{\psi'(\xi)} < 2,$$

and

$$(1.8) \quad \int_1^\infty \sqrt{\frac{\psi'(\xi^2)}{\psi(\xi^2)}} e^{-t\psi(\xi^2)} d\xi < \infty,$$

then

$$(1.9) \quad \mathbf{P}_x(\tau_{(0,\infty)} > t) = \frac{2}{\pi} \int_0^\infty \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} e^{-t\psi(\lambda^2)} F_\lambda(x) d\lambda \quad \text{for } x > 0.$$

If $\sqrt{\psi(\xi^2)\psi'(\xi^2)}e^{-t\psi(\xi^2)}$ is integrable in $\xi > 1$ for all $t > 0$, then furthermore

$$(1.10) \quad \mathbf{P}_x(\tau_{(0,\infty)} \in dt) = \frac{2}{\pi} \left(\int_0^\infty \sqrt{\psi'(\lambda^2)\psi(\lambda^2)} e^{-t\psi(\lambda^2)} F_\lambda(x) d\lambda \right) dt$$

for $x, t > 0$.

Remark 1.9. Theorem 1.8 in full generality is proved in the recent preprint [56]. In this article we provide a much simpler proof, under a more restrictive condition

$$(1.11) \quad \limsup_{\xi \rightarrow 0^+} \frac{\xi |\psi''(\xi)|}{\psi'(\xi)} < 1, \quad \limsup_{\xi \rightarrow \infty} \frac{\xi |\psi''(\xi)|}{\psi'(\xi)} < 1.$$

This condition implies both (1.7) and (1.8), as well as integrability of $\sqrt{\psi(\xi^2)\psi'(\xi^2)}e^{-t\psi(\xi^2)}$ (see the proof of the theorem). The proof given below relies on Theorem 1.3. Noteworthy, many interesting examples, including symmetric α -stable processes and relativistic α -stable processes, satisfy (1.11). By Corollary 4.26, condition (1.11) is satisfied when ψ is regularly varying of positive order at 0 and at ∞ . \square

It should be emphasized that the integrand in (1.6) has two oscillatory factors, and cancellations in the integral are essential. Hence, formula (1.6) is problematic for numerical computations. For the Cauchy process, a significant simplification of (1.6) is possible (see Theorem 4 in [49]). It is an open problem whether formula (1.6) can be simplified in the more general case, even for symmetric α -stable processes for general $\alpha \in (0, 2]$.

Cancellation is of less importance in Theorem 1.8; see [56], where (1.10) is used to obtain estimates for the density function of the distribution of $\tau_{(0,\infty)}$.

Since $\xi |\psi''(\xi)|/\psi'(\xi) < 2$ for all $\xi > 0$, the supremum in (1.7) is always not greater than 2 (see Preliminaries). The assumptions (1.7) and (1.8) are

rather mild regularity and growth conditions for $\psi(\xi)$; examples are given in Section 6.

Remark 1.10. By symmetry, the first exit time $\tau_{(0,\infty)}$ for X_t starting at $x > 0$ has the same distribution as the first passage time τ_x through a barrier at x for X_t starting at 0. On the other hand, first passage times are related to the supremum functional:

$$\mathbf{P}_0 \left(\sup_{s \in [0,t]} X_s < x \right) = \mathbf{P}_0(\tau_x > t) = \mathbf{P}_x(\tau_{(0,\infty)} > t).$$

Hence, Theorem 1.8 gives, in a rather general setting, an explicit expression for the distribution of first passage times and the supremum functional, which are fundamental objects in fluctuation theory of Lévy processes. Noteworthy, the double Laplace transform (in t and x) of the distribution of τ_x is known for general Lévy processes since 1957 (Theorem 1 in [4]). Nevertheless, explicit formulae for $\mathbf{P}_x(\tau_{(0,\infty)} > t)$ are known only in some special cases; see [24, 32, 35, 51, 52] for some recent developments in this area. Also, a formula for the single Laplace transform (in t) of $\mathbf{P}(\tau_x > t)$ for a large class of symmetric Lévy processes was obtained only very recently in [55]. \square

Theorems 1.1 and 1.3 can be applied to certain systems of PDEs, related to traces of two-dimensional diffusions, as discussed in Section 7. Furthermore, there are at least four recent preprints exploiting the results of the present article. An application of Theorem 1.1 to the spectral theory of symmetric α -stable process in the interval can be found in [54]. The case of the relativistic 1-stable process in half-line and interval is studied in detail in [38]. This application may be of particular interest in quantum physics; see [26, 29, 58] for related research. In [27], Theorems 1.1 and 1.3 are used to obtain refined semi-classical asymptotics for eigenvalues of higher-dimensional isotropic α -stable processes in domains. Finally, the proof of Theorems 1.3 and 1.8 in full generality, as well as their application to a detailed analysis of first passage times, can be found in [56].

We conclude the introduction with a brief description of the structure of the article. The Preliminaries section contains definitions and standard properties used in the article, and auxiliary lemmas. It includes some background on distribution theory, Lévy processes and their generators, subordinate Brownian motions and complete Bernstein functions. In Section 3 a certain transformation related to the Wiener-Hopf factorization is studied. Next, in Section 4, we derive the formula for F_λ and prove Theorem 1.1. Theorems 1.3, 1.7 and 1.8 are proved in Section 5. Examples, including symmetric α -stable processes and relativistic α -stable processes, are studied in detail in Section 6. In Section 7 we give an application of our results to systems of PDEs (Theorem 7.1).

2. PRELIMINARIES

2.1. Distribution theory. In the theory of partial differential equations, it is a common case that one first finds a weak solution to a problem, and

then, by showing that the solution is sufficiently regular, one argues that it is in fact a strong solution. This approach will be used to prove Theorem 1.1: first we find a distributional eigenfunction of $\mathcal{A}_{(0,\infty)}$, and then we show that in fact it belongs to the domain of $\mathcal{A}_{(0,\infty)}$. Our argument relies heavily on Fourier methods, and the operators involved are typically non-local, so the required background on distribution theory is slightly different from the one used in partial differential equations. It should be emphasized that the use of distributions could be avoided, at least partially, at the price of less clear exposition. However, the language of distribution theory seems to be well-suited for problems involving generators of killed Lévy processes.

Let \mathcal{S} denote the class of Schwartz functions in \mathbf{R}^d ($d = 1, 2, \dots$), and let \mathcal{S}' be the space of tempered distributions in \mathbf{R}^d . If $\varphi \in \mathcal{S}$ and $F \in \mathcal{S}'$, we write $\langle F, \varphi \rangle$ for the value of F at φ . Below we recall some well-known properties of tempered distributions; for a detailed exposition of the theory, see e.g. [71].

If $\varphi \in \mathcal{S}$, the Fourier transform of φ is $\mathcal{F}\varphi(\xi) = \int_{\mathbf{R}^d} e^{i\xi \cdot x} \varphi(x) dx$. For $F \in \mathcal{S}'$, $\mathcal{F}F$ is the tempered distribution satisfying $\langle \mathcal{F}F, \varphi \rangle = \langle F, \mathcal{F}\varphi \rangle$.

The convolution of $\varphi_1, \varphi_2 \in \mathcal{S}$ is defined in the usual way, $\varphi_1 * \varphi_2(x) = \int_{\mathbf{R}^d} \varphi_1(y) \varphi_2(x - y) dy$. When $\varphi \in \mathcal{S}$ and $F \in \mathcal{S}'$, then $F * \varphi$ is an infinitely smooth function, defined by

$$F * \varphi(x) = \langle F, \varphi_x \rangle, \quad \text{where} \quad \varphi_x(y) = \varphi(x - y).$$

The convolution of two distributions is not well-defined in general. Suppose that $F_1, F_2 \in \mathcal{S}'$. We say that F_1 and F_2 are \mathcal{S}' -convolvable if for all $\varphi_1, \varphi_2 \in \mathcal{S}$, the functions $F_1 * \varphi_1$ and $F_2 * \varphi_2$ are convolvable in the usual sense, i.e. the integral $\int_{\mathbf{R}^d} (F_1 * \varphi_1)(y) (F_2 * \varphi_2)(x - y) dy$ exists for all x . When this is the case, the \mathcal{S}' -convolution $F_1 * F_2$ is the unique distribution F satisfying $F * (\varphi_1 * \varphi_2) = (F_1 * \varphi_1) * (F_2 * \varphi_2)$ for $\varphi_1, \varphi_2 \in \mathcal{S}$. Note that there are other non-equivalent definitions of the convolution of distributions, and \mathcal{S}' -convolution is often denoted as $F_1 \otimes F_2$; for the discussion of various notions of convolvability, the reader is referred to [22, 71].

Recall that the *support* of a distribution F is the smallest closed set $\text{supp } F$ with the property that $\langle F, \varphi \rangle = 0$ for all $\varphi \in \mathcal{S}$ such that $\varphi(x) = 0$ for $x \in \text{supp } F$. If any of the tempered distributions F_1, F_2 has compact support, or (in the one-dimensional case) if both F_1 and F_2 are supported in $[0, \infty)$, then F_1 and F_2 are automatically \mathcal{S}' -convolvable.

It is well known that the distributions $(F_1 * F_2) * F_3$ and $F_1 * (F_2 * F_3)$ need not be equal; however, if the pairs (F_1, F_2) and (F_2, F_3) are \mathcal{S}' -convolvable, and furthermore the functions $F_1 * \varphi_1$, $F_2 * \varphi_2$, and $F_3 * \varphi_3$ are convolvable (that is, $(F_1 * \varphi_1)(y) (F_2 * \varphi_2)(z) (F_3 * \varphi_3)(x - y - z)$ is integrable in y, z for all $x \in \mathbf{R}^d$) for any $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{S}$, then the \mathcal{S}' -convolution of F_1, F_2 and F_3 is associative; see [71], Section 4.2.8.

Any \mathcal{S}' -convolvable distributions F_1, F_2 satisfy the *exchange formula* $\mathcal{F}(F_1 * F_2) = \mathcal{F}F_1 \cdot \mathcal{F}F_2$, where the multiplication of distributions $\mathcal{F}F_1$ and $\mathcal{F}F_2$ extends standard multiplication of functions in an appropriate manner, see [34, 71]. Since we only use the exchange formula when $\mathcal{F}F_1$ and $\mathcal{F}F_2$ are genuine functions, or when $\mathcal{F}F_1$ is a measure and $\mathcal{F}F_2$ is a function,

we do not discuss the notion of multiplication for general distributions and refer the interested reader to [39, 65, 71].

In the one-dimensional case $d = 1$, the Laplace transform can be defined for tempered distributions. If $\varphi \in \mathcal{S}$ is supported in $[0, \infty)$, then the Laplace transform of φ is denoted by $\mathcal{L}\varphi(\xi) = \int_{\mathbf{R}} e^{-\xi x} \varphi(x) dx$ ($\operatorname{Re} \xi \geq 0$). This is a holomorphic function of ξ in the right complex half-plane $\operatorname{Re} \xi > 0$, continuous at the boundary. Clearly, $\mathcal{F}\varphi(\xi) = \mathcal{L}\varphi(-i\xi)$. If $F \in \mathcal{S}'$ is supported in $[0, \infty)$, the Laplace transform of F is defined for $\xi \in \mathbf{C}$ with $\operatorname{Re} \xi > 0$ by

$$\mathcal{L}F(\xi) = \langle F, e_\xi h \rangle,$$

where $e_\xi(x) = e^{-\xi x}$ and h is any infinitely smooth function such that $h(x) = 1$ for $x \geq 0$ and $h(x) = 0$ for $x \leq -1$. This definition does not depend on the choice of h (see formula (9.1.4) in [71]). The above definition of $\mathcal{L}F$ extends the usual definition of the Laplace transform of signed measures on $[0, \infty)$.

Note that while the Fourier transform of a distribution is again a distribution, the Laplace transform is always a (holomorphic) function. For a fixed $t > 0$, the function $\mathcal{L}F(t - is)$ is the Fourier transform of the distribution $e_t F$ (where again $e_t(x) = e^{-tx}$), and as $t \rightarrow 0^+$, the distributions $\mathcal{L}F(t - is)$ converge to $\mathcal{F}F$ in \mathcal{S}' . The exchange formula holds also for the Laplace transform: if both $F_1, F_2 \in \mathcal{S}'$ are supported in $[0, \infty)$, then they are \mathcal{S}' -convolvable and $\mathcal{L}(F_1 * F_2)(\xi) = \mathcal{L}F_1(\xi)\mathcal{L}F_2(\xi)$ for all ξ with $\operatorname{Re} \xi > 0$.

We say that two distributions F_1, F_2 restricted to an open set D are equal if $\langle F_1, \varphi \rangle = \langle F_2, \varphi \rangle$ for any $\varphi \in \mathcal{S}$ vanishing in $\mathbf{R}^d \setminus D$. Equivalently: $F_1 - F_2$ is supported in $\mathbf{R}^d \setminus D$.

2.2. Transition semigroups and generators. Below we recall and extend some standard definitions and properties, which can be found, for example, in [1, 5, 64]. In this article we are only concerned with one-dimensional subordinate Brownian motions. However, Lemma 2.10 below might be of interest for general Lévy processes. For this reason, in this and the next subsection, we work with the general case.

Let X_t be a Lévy process in \mathbf{R}^d . We write \mathbf{P}_x and \mathbf{E}_x for the probability and expectation for the process X_t which starts at $x \in \mathbf{R}^d$. The process X_t is completely determined by its Lévy-Khintchine exponent Ψ : for $\xi \in \mathbf{R}^d$ we have

$$(2.1) \quad \mathbf{E}_0 e^{i\xi \cdot X_t} = e^{-t\Psi(\xi)},$$

where

$$(2.2) \quad \Psi(\xi) = \beta\xi \cdot \xi - i\gamma \cdot \xi + \int_{\mathbf{R}^d} (1 - e^{i\xi \cdot z} + \mathbf{1}_{|z| \leq 1} i\xi \cdot z) \nu(dz).$$

Here β is a nonnegative definite $d \times d$ matrix (the *diffusion coefficient*), γ is a vector in \mathbf{R}^d (the *drift*), and ν is the *Lévy measure* of X_t : a Radon measure on $\mathbf{R}^d \setminus \{0\}$ such that $\int_{\mathbf{R}^d} \min(1, |z|^2) \nu(dz) < \infty$. If ν is absolutely continuous, we denote its density function with the same symbol $\nu(z)$.

The *transition operators* of X_t are defined by

$$P_t f(x) = \mathbf{E}_x f(X_t) = \int_{\mathbf{R}^d} f(y) \mathbf{P}_x(X_t \in dy), \quad t > 0, x \in \mathbf{R},$$

whenever the integral is absolutely convergent. Each P_t is a convolution operator, with convolution kernel given by the distribution of $(-X_t)$ under \mathbf{P}_0 . By the Lévy-Khintchine formula (2.1),

$$(2.3) \quad P_t e_\xi = e^{-t\Psi(\xi)} e_\xi \quad \text{for} \quad e_\xi(x) = e^{i\xi \cdot x}.$$

Furthermore, for $\varphi \in \mathcal{S}$,

$$(2.4) \quad \mathcal{F}P_t \varphi(\xi) = e^{-t\overline{\Psi(\xi)}} \mathcal{F}\varphi(\xi), \quad \xi \in \mathbf{R}^d;$$

see Theorem 3.3.3 in [1] (note that in [1], Fourier transform is defined with $e^{-i\xi \cdot x}$ instead of $e^{i\xi \cdot x}$). The operators P_t form a contraction semigroup on each of the Lebesgue spaces $L^p(\mathbf{R}^d)$ ($p \in [1, \infty]$), on the space $C_b(\mathbf{R}^d)$ of bounded continuous functions, and on the space $C_0(\mathbf{R}^d)$ of continuous functions vanishing at infinity (all equipped with usual norms). On $L^p(\mathbf{R}^d)$ ($p \in [1, \infty)$) and on $C_0(\mathbf{R}^d)$, this semigroup is strongly continuous (see Section 3.4 in [1]). The generator of each of these semigroups is denoted by the same symbol \mathcal{A} , and we write $\mathcal{D}(\mathcal{A}; \mathcal{X})$ for the corresponding domain, where \mathcal{X} is either $L^p(\mathbf{R}^d)$ ($p \in [1, \infty]$), $C_b(\mathbf{R}^d)$ or $C_0(\mathbf{R}^d)$. More precisely, $f \in \mathcal{D}(\mathcal{A}; \mathcal{X})$ if $f \in \mathcal{X}$ and the limit

$$\mathcal{A}f = \lim_{t \rightarrow 0^+} \frac{P_t f - f}{t}$$

exists in the topology of \mathcal{X} . Note that the limit, if it exists, does not depend on the choice of \mathcal{X} (apart from the fact that in $C_0(\mathbf{R}^d)$ and $C_b(\mathbf{R}^d)$ it is defined pointwise, while in $L^p(\mathbf{R}^d)$ only up to a set of zero measure). Therefore, using a single symbol \mathcal{A} for operators acting on different domains $\mathcal{D}(\mathcal{A}; \mathcal{X})$ causes no confusion. The Schwartz class \mathcal{S} is a core of \mathcal{A} on each of the spaces $L^p(\mathbf{R}^d)$ ($p \in [1, \infty)$) and $C_0(\mathbf{R}^d)$, and by (2.4), for $\varphi \in \mathcal{S}$,

$$(2.5) \quad \mathcal{F}\mathcal{A}\varphi(\xi) = -\overline{\Psi(\xi)} \mathcal{F}\varphi(\xi), \quad \xi \in \mathbf{R}.$$

We abbreviate, for example, $\mathcal{D}(\mathcal{A}; L^p(\mathbf{R}^d))$ to $\mathcal{D}(\mathcal{A}; L^p)$.

Remark 2.1. The spectral theory of P_t and \mathcal{A} is very simple, thanks to the Lévy Khintchine formula (2.3). Indeed, the function $e^{i\xi \cdot x}$ is the eigenfunction of P_t , with eigenvalue $e^{-t\Psi(\xi)}$. Hence, $e^{i\xi \cdot x}$ belongs to $\mathcal{D}(\mathcal{A}; L^\infty)$, and it is an eigenfunction of \mathcal{A} with eigenvalue $-\Psi(\xi)$. The generalized eigenfunction expansion of P_t and \mathcal{A} is given by (2.4) and (2.5); the Fourier transform plays the same role as the integral transform Π in Theorem 1.3.

If X_t is symmetric, then $\Psi(\xi) = \Psi(-\xi)$ is real. Since $\sin(\xi \cdot x + \vartheta)$ is a linear combination of $e^{i\xi \cdot x}$ and $e^{i(-\xi) \cdot x}$, it is also the eigenfunction of P_t and \mathcal{A} . Note that the eigenfunctions F_λ in Theorem 1.1 behave asymptotically as $\sin(\lambda x + \vartheta_\lambda)$ as $x \rightarrow \infty$. \square

When $\varphi \in \mathcal{S}$, then the generator of X_t can be written in the form

$$(2.6) \quad \begin{aligned} \mathcal{A}\varphi(x) &= \beta \nabla \varphi(x) \cdot \nabla \varphi(x) + \gamma \cdot \nabla \varphi(x) \\ &+ \int_{\mathbf{R}^d} (\varphi(x+z) - \varphi(x) - \mathbf{1}_{|z| \leq 1} z \cdot \nabla \varphi(x)) \nu(dz), \end{aligned}$$

where β , γ and ν are as in (2.2) (Theorem 31.5 in [64]). Furthermore,

$$(2.7) \quad \|\mathcal{A}\varphi\|_{L^\infty(\mathbf{R}^d)} \leq C \|\varphi\|_{C_b^2(\mathbf{R}^d)}, \quad \varphi \in \mathcal{S},$$

where C depends only on the process X_t , and $\|\varphi\|_{C_b^2(\mathbf{R}^d)}$ is the maximum of $L^\infty(\mathbf{R}^d)$ norms of φ and its first and second partial derivatives (see the proof of Theorem 31.5 in [64]).

We need the following extension of (2.6). For $C_0(\mathbf{R}^d)$ instead of $C_b(\mathbf{R}^d)$, this is well-known, see Theorem 31.5 in [64]. The statement given below is, however, difficult to find in the literature, so we provide a proof. Let $C_b^\infty(\mathbf{R}^d)$ be the class of functions F such that F and all partial derivatives of F (of all orders) belong to $C_b(\mathbf{R}^d)$.

Lemma 2.2. *If $F \in C_b^\infty(\mathbf{R}^d)$, then $F \in \mathcal{D}(\mathcal{A}; C_b)$ and*

$$(2.8) \quad \begin{aligned} \mathcal{A}F(x) &= \beta \nabla F(x) \cdot \nabla F(x) + \gamma \cdot \nabla F(x) \\ &+ \int_{\mathbf{R}^d} (F(x+z) - F(x) - \mathbf{1}_{|z| \leq 1} z \cdot \nabla F(x)) \nu(dz). \end{aligned}$$

Remark 2.3. Note that in contrast to the $C_0(\mathbf{R}^d)$ case, F may fail to belong to $\mathcal{D}(\mathcal{A}; C_b)$ when F and its first and second partial derivatives are in $C_b(\mathbf{R}^d)$. A simple counterexample can be easily constructed for one-dimensional Brownian motion: if $F, F', F'' \in C_b(\mathbf{R})$, but F'' is not *uniformly* continuous, then the convergence of $(P_t f - f)/t$ may fail to be uniform. We omit the details. \square

Proof of Lemma 2.2. Let $\|f\|_{C_b^k(\mathbf{R}^d)}$ denote the maximum of $L^\infty(\mathbf{R}^d)$ norms of f and its partial derivatives of order not greater than k , and denote $M = \|F\|_{C_b^4(\mathbf{R}^d)}$. There is a sequence $\varphi_n \in \mathcal{S}$ such that φ_n and its first and second partial derivatives converge locally uniformly to F and the corresponding first and second partial derivatives of F , and furthermore $\|\varphi_n\|_{C_b^4(\mathbf{R}^d)} \leq CM$ (with C depending only on the dimension d). For $f \in \mathcal{D}(\mathcal{A}; L^\infty)$ and $x \in \mathbf{R}^d$, we have (see [25])

$$(2.9) \quad P_t f(x) - f(x) = \int_0^t P_s \mathcal{A}f(x) ds = \int_0^t \mathcal{A}P_s f(x) ds.$$

In particular, $\|P_t f - f\|_{L^\infty(\mathbf{R}^d)} \leq t \|\mathcal{A}f\|_{L^\infty(\mathbf{R}^d)}$. Since $\varphi_n \in \mathcal{S} \subseteq \mathcal{D}(\mathcal{A}; C_0)$, by (2.9) we have

$$\begin{aligned} \left| \frac{P_t \varphi_n(x) - \varphi_n(x)}{t} - \mathcal{A}\varphi_n(x) \right| &= \left| \frac{1}{t} \int_0^t \mathcal{A}P_s \varphi_n(x) ds - \mathcal{A}\varphi_n(x) \right| \\ &\leq \frac{1}{t} \int_0^t |\mathcal{A}(P_s \varphi_n - \varphi_n)(x)| ds. \end{aligned}$$

By (2.7),

$$\left| \frac{P_t \varphi_n(x) - \varphi_n(x)}{t} - \mathcal{A} \varphi_n(x) \right| \leq \frac{1}{t} \int_0^t \|P_s \varphi_n - \varphi_n\|_{C_b^2(\mathbf{R}^d)} ds.$$

Since P_s and \mathcal{A} commute with partial differential operators on \mathcal{S} , by (2.9),

$$\left| \frac{P_t \varphi_n(x) - \varphi_n(x)}{t} - \mathcal{A} \varphi_n(x) \right| \leq \frac{1}{t} \int_0^t s \|\mathcal{A} \varphi_n\|_{C_b^2(\mathbf{R}^d)} ds = \frac{t}{2} \|\mathcal{A} \varphi_n\|_{C_b^2(\mathbf{R}^d)}.$$

Finally, by (2.7),

$$\left| \frac{P_t \varphi_n(x) - \varphi_n(x)}{t} - \mathcal{A} \varphi_n(x) \right| \leq \frac{t}{2} \|\varphi_n\|_{C_b^4(\mathbf{R}^d)} \leq \frac{CtM}{2}.$$

As $n \rightarrow \infty$, we have $\varphi_n(x) \rightarrow F(x)$ and $P_t \varphi_n(x) \rightarrow P_t F(x)$ (by dominated convergence). By (2.6), Taylor's theorem and dominated convergence, $\mathcal{A} \varphi_n(x)$ converges to the right-hand side of (2.8), which we denote by $G(x)$, and the convergence is locally uniform in $x \in \mathbf{R}^d$. It follows that

$$\left| \frac{P_t F(x) - F(x)}{t} - G(x) \right| \leq \frac{CtM}{2}.$$

Since $x \in \mathbf{R}^d$ was arbitrary, we have $F \in \mathcal{D}(\mathcal{A}; C_b)$ and $\mathcal{A}F(x) = G(x)$, as desired. \square

Informally, the generator \mathcal{A} is also a convolution operator, but the convolution kernel is a tempered distribution. We give this a precise meaning; see also [5, 64], and [8] for the case of symmetric α -stable processes.

Definition 2.4. The *distributional generator* of X_t is the tempered distribution $A \in \mathcal{S}'$ defined by the formula

$$\langle A, \varphi \rangle = \mathcal{A} \varphi(0), \quad \varphi \in \mathcal{S}.$$

Let $\check{\varphi}(x) = \varphi(-x)$ for $\varphi \in \mathcal{S}$. We define \check{A} (the *reflection* of A) by the formula $\langle \check{A}, \varphi \rangle = \langle A, \check{\varphi} \rangle$. Hence, $\langle \check{A}, \varphi \rangle = \check{A} * \varphi(0)$ and $\mathcal{A} \varphi = \check{A} * \varphi$ for $\varphi \in \mathcal{S}$ (see [71]). By (2.3), $\mathcal{F} \check{A}(\xi) = -\Psi(\xi)$ and $\mathcal{F} A(\xi) = -\Psi(\xi)$ for $\xi \in \mathbf{R}^d$.

Proposition 2.5. When $F \in \mathcal{D}(\mathcal{A}; L^\infty)$, then $\mathcal{A}F = \check{A} * F$.

Proof. We have, by the definition of \mathcal{S}' -convolution,

$$(\check{A} * F) * \varphi_1 * \varphi_2 = (\check{A} * \varphi_1) * (F * \varphi_2) = (\mathcal{A} \varphi_1) * F * \varphi_2.$$

Recall that $\mathcal{S} \subseteq \mathcal{D}(\mathcal{A}; L^1)$. Hence, $\mathcal{A} \varphi_1$ is the $L^1(\mathbf{R}^d)$ limit of $(P_t \varphi_1 - \varphi_1)/t$. Since φ_1, φ_2 and the convolution kernel of P_t are integrable, and F is bounded, by Fubini we have

$$(P_t \varphi_1 - \varphi_1) * F * \varphi_2 = (P_t F - F) * \varphi_1 * \varphi_2.$$

By dominated convergence,

$$(\check{A} * F) * \varphi_1 * \varphi_2 = \lim_{t \rightarrow 0^+} \frac{(P_t F - F) * \varphi_1 * \varphi_2}{t} = (\mathcal{A}F) * \varphi_1 * \varphi_2,$$

as desired. \square

Corollary 2.6. *If $F \in C_b^\infty(\mathbf{R}^d)$ (as in Lemma 2.2), then $\mathcal{A}F \in C_b^\infty(\mathbf{R}^d)$, and \mathcal{A} commutes with partial differential operators on $C_b^\infty(\mathbf{R}^d)$.*

Proof. Let D be a partial derivative operator (of arbitrary order). By Proposition 2.5, we have $D\mathcal{A}F = D(\check{A} * F) = \check{A} * (DF) = \mathcal{A}DF$. \square

When X_t is one-dimensional and symmetric, then (2.2) takes the form

$$(2.10) \quad \Psi(\xi) = \beta\xi^2 + \int_{-\infty}^{\infty} (1 - \cos(\xi z))\nu(dz), \quad \xi \in \mathbf{R},$$

where $\beta \geq 0$ and $\int_{\mathbf{R}} \min(1, z^2)\nu(dz) < \infty$. For $f \in C_b^\infty(\mathbf{R}^d)$ (as in Lemma 2.2), we have

$$(2.11) \quad \mathcal{A}f(x) = \beta f''(x) + \text{pv} \int_{\mathbf{R}} (f(x+z) - f(x))\nu(dz), \quad x \in \mathbf{R}.$$

Here $\text{pv} \int$ denotes the Cauchy principal value integral:

$$\text{pv} \int_{\mathbf{R}} (f(x+z) - f(x))\nu(dz) = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbf{R} \setminus (-\varepsilon, \varepsilon)} (f(x+z) - f(x))\nu(dz).$$

Of course, when X_t is symmetric, then $\check{A} = A$.

2.3. Killed process and its generator. The main references for the notion of a killed process (or part of a Markov process) are [7, 25], where general strong Markov processes are studied. Here we consider a Lévy process X_t in \mathbf{R}^d . Let τ_D be the time of first exit from D , $\tau_D = \inf \{t \geq 0 : X_t \notin D\}$. The *killed process* X_t^D is a strong Markov process in D with lifetime τ_D , such that $X_t^D = X_t$ for all $t < \tau_D$. The transition operators P_t^D , the generator \mathcal{A}_D , and domains $\mathcal{D}(\mathcal{A}_D; \mathcal{X})$ corresponding to the killed process X_t^D are defined in the same way as for the free process X_t , that is,

$$P_t^D f(x) = \mathbf{E}_x f(X_t^D) = \mathbf{E}_x (f(X_t) \mathbf{1}_{t < \tau_D}),$$

and

$$\mathcal{A}_D f = \lim_{t \rightarrow 0^+} \frac{P_t^D f - f}{t},$$

whenever the limit exists in the topology of the function space \mathcal{X} ; in this case, we write $f \in \mathcal{D}(\mathcal{A}_D; \mathcal{X})$. Again we abbreviate, for example, $\mathcal{D}(\mathcal{A}_D; L^p(D))$ to $\mathcal{D}(\mathcal{A}_D; L^p)$.

When X_t is the Brownian motion, then \mathcal{A} is the Laplace operator in \mathbf{R}^d , and \mathcal{A}_D is the Laplace operator in D with Dirichlet boundary condition. Even in this case the relation between \mathcal{A}_D and \mathcal{A} is very delicate. On the other hand, this relation is crucial for the proof of Theorem 1.1. Hence, we now discuss this topic in more detail.

Suppose first that X_t is a compound Poisson process: $\beta = 0$, $\gamma = 0$ and ν is a finite measure. In this case, the situation is clear. The following result seems to be well-known to specialists, but it is difficult to find in literature (cf. Example 3.3.7 in [1] for the case $D = \mathbf{R}^d$).

Lemma 2.7. *Let X_t be a compound Poisson process in \mathbf{R}^d , with Lévy measure ν , and let $p \in [1, \infty]$. Then the semigroup P_t^D is strongly continuous on $L^p(D)$, the generator \mathcal{A}_D is a bounded operator on each $L^p(D)$, and*

$$\mathcal{A}_D f(x) = \mathbf{1}_D(x) \mathcal{A} f(x) = \mathbf{1}_D(x) \int_{\mathbf{R}^d} (f(x+z) - f(x)) \nu(dz)$$

for all $f \in L^p(D)$, with the usual extension $f(x) = 0$ for $x \notin D$.

Proof. Denote by $M = \nu(\mathbf{R}^d \setminus \{0\})$ the intensity of jumps of X_t , and let $0 < \tau_1 < \tau_2 < \dots$ be the sequence of (random) times when X_t jumps. Let f be in some $L^p(D)$ ($p \in [1, \infty]$), and let $x \in D$ be the starting point of X_t . Note that

$$\begin{aligned} P_t^D f(x) &= \mathbf{E}_x(f(X_t) \mathbf{1}_{t < \tau_D}) = \mathbf{E}_x(f(X_t) \mathbf{1}_{t < \tau_1}) \\ &\quad + \mathbf{E}_x(f(X_t) \mathbf{1}_{\tau_1 \leq t < \tau_2}) \\ &\quad + \mathbf{E}_x(f(X_t) \mathbf{1}_{\tau_2 \leq t < \tau_D}); \end{aligned}$$

indeed, in the first summand $t < \tau_1$ implies that $t < \tau_D$, and in the second one, when $\tau_1 \leq t < \tau_2$ and $t \geq \tau_D$, then $f(X_t) = 0$. We have $f(X_t) = f(x)$ in the first summand, and in the second one, $f(X_t) = f(X_{\tau_1})$ is independent of (τ_1, τ_2) . Furthermore, $\mathbf{P}_x(X_{\tau_1} \in x + dz) = M^{-1} \nu(dz)$. Hence, if $\check{\nu}(E) = \nu(-E)$, we have

$$\mathbf{E}_x f(X_{\tau_1}) = \int_{\mathbf{R}^d} f(x+z) \nu(dz) = \int_{\mathbf{R}^d} f(x-z) \check{\nu}(dz) = f * \check{\nu}(x).$$

It follows that,

$$\begin{aligned} P_t^D f(x) &= f(x) \mathbf{P}_x(t < \tau_1) \\ &\quad + M^{-1} f * \check{\nu}(x) \mathbf{P}_x(\tau_1 \leq t < \tau_2) \\ &\quad + \mathbf{E}_x(f(X_t) \mathbf{1}_{\tau_2 \leq t < \tau_D}), \end{aligned}$$

and so, finally,

$$\begin{aligned} (2.12) \quad & \left| \frac{P_t^D f(x) - f(x)}{t} - (f * \check{\nu}(x) - M f(x)) \right| \\ & \leq \left| \frac{\mathbf{P}_x(t < \tau_1) - 1 + Mt}{t} f(x) \right| + \left| \frac{\mathbf{P}_x(\tau_1 \leq t < \tau_2) - Mt}{Mt} f * \check{\nu}(x) \right| \\ & \quad + \left| \frac{\mathbf{E}_x(f(X_t) \mathbf{1}_{\tau_2 \leq t < \tau_D})}{t} \right|. \end{aligned}$$

The proof will be complete once we show that each of the summands on the right-hand side converges to 0 in $L^p(D)$.

Since τ_1, τ_2, \dots are increase times of a Poisson process, we have

$$\mathbf{P}_x(\tau_k \leq t < \tau_{k+1}) = \frac{(Mt)^k}{k!} e^{-Mt}, \quad t > 0, k = 0, 1, \dots,$$

where $\tau_0 = 0$. In particular, $|\mathbf{P}_x(t < \tau_1) - 1 + Mt| = |e^{-Mt} - 1 + Mt| \leq (Mt)^2/2$ and $|\mathbf{P}_x(\tau_1 \leq t < \tau_2) - Mt| = |Mte^{-Mt} - Mt| \leq (Mt)^2$, which shows

that the first two summands in (2.12) converge to 0 in $L^p(\mathbf{R}^d)$. Furthermore,

$$\begin{aligned}
|\mathbf{E}_x(f(X_t)\mathbf{1}_{\tau_2 \leq t < \tau_D})| &\leq \mathbf{E}_x(|f(X_t)|\mathbf{1}_{\tau_2 \leq t}) \\
&= \sum_{k=2}^{\infty} \mathbf{E}_x(|f(X_{\tau_k})|\mathbf{1}_{\tau_k \leq t < \tau_{k+1}}) \\
&= \sum_{k=2}^{\infty} \mathbf{E}_x|f(X_{\tau_k})|\mathbf{P}_x(\tau_k \leq t < \tau_{k+1}) \\
&= \sum_{k=2}^{\infty} \frac{|f| * (\check{\nu}^{*k})(x)}{M^k} \frac{(Mt)^k}{k!} e^{-Mt}.
\end{aligned}$$

Since the total mass of $\check{\nu}^{*k}$ is M^k , we conclude that

$$\begin{aligned}
\|\mathbf{E}_x(|f|(X_t)\mathbf{1}_{\tau_2 \leq t < \tau_D})\|_{L^p(D)} &\leq \left\| \sum_{k=2}^{\infty} \frac{f * (\check{\nu}^{*k})}{M^k} \frac{(Mt)^k}{k!} e^{-Mt} \right\|_{L^p(\mathbf{R}^d)} \\
&\leq \sum_{k=2}^{\infty} \|f\|_{L^p(\mathbf{R}^d)} \frac{(Mt)^k}{k!} e^{-Mt} \\
&= \|f\|_{L^p(\mathbf{R}^d)} (1 - e^{-Mt} - Mte^{-Mt}),
\end{aligned}$$

which is of order t^2 as $t \rightarrow 0^+$. Hence, also the third summand in (2.12) converges to 0. \square

When X_t is not a compound Poisson process, then the generators of X_t and X_t^D are unbounded operators, and the relation between the domains of \mathcal{A} and \mathcal{A}_D is much less obvious. A point $x \in \partial D$ is said to be a *regular boundary point of D* if $\inf\{t > 0 : X_t \notin D\} = 0$ a.s. with respect to \mathbf{P}_x (note that here the inequality $t > 0$ is strict, while $t \geq 0$ is used in the definition of τ_D). Let $C_0(D)$ denote the space of $C_0(\mathbf{R}^d)$ functions vanishing in $\mathbf{R}^d \setminus D$. If every $x \in \partial D$ is a regular boundary point and X_t has the strong Feller property (that is, P_t maps $L^\infty(\mathbf{R}^d)$ to $C_b(\mathbf{R}^d)$), then the operators P_t^D form a strongly continuous contraction semigroup on $C_0(D)$ (that is, X_t^D has the Feller property; see [18]). Note that if X_t is a symmetric Lévy process in \mathbf{R} (and not a compound Poisson process) and $D = (0, \infty)$, then 0, the only boundary point of D , is regular (see Theorem 47.5 in [64]).

We have the following fundamental result due to Dynkin.

Definition 2.8 (Dynkin characteristic operator, [25]). Let $x \in \mathbf{R}^d$ and denote by $\tau_{x,\varepsilon}$ the first exit time from the ball centered at x and with radius ε . If the limit

$$\mathfrak{A}_x f = \lim_{\varepsilon \rightarrow 0^+} \frac{\mathbf{E}_x f(X_{\tau_{x,\varepsilon}}) - f(x)}{\mathbf{E}_x \tau_{x,\varepsilon}}$$

exists, we write $f \in \mathcal{D}(\mathfrak{A}_x)$. If $f \in \mathcal{D}(\mathfrak{A}_x)$ for all $x \in D$, we write $f \in \mathcal{D}(\mathfrak{A}_D)$, and we define $\mathfrak{A}_D f(x) = \mathfrak{A}_x f$.

Lemma 2.9 (Theorem 5.5 in [25] for Lévy processes). *Suppose that X_t is a Lévy process in \mathbf{R}^d which is a strong Feller process. Suppose furthermore that $D \subseteq \mathbf{R}^d$ is open, and every boundary point of D is regular. Let $f \in$*

$C_0(D)$. Then $f \in \mathcal{D}(\mathcal{A}_D; C_0)$ if and only if $f \in \mathcal{D}(\mathfrak{A}_D)$ and $\mathfrak{A}_D f \in C_0(D)$. In this case, $\mathcal{A}_D f(x) = \mathfrak{A}_D f(x)$ for $x \in D$. \square

The above lemma states that, in a sense, the operators \mathcal{A}_D and \mathcal{A} have the same *pointwise* definition, but their domains are different. This can be made even more explicit using the language of distribution theory, as we now describe. Recall that A is the distributional generator, defined in Definition 2.4, and \check{A} is the reflection of A .

Lemma 2.10. *Suppose that X_t is a Lévy process in \mathbf{R}^d which is a strong Feller process. Suppose furthermore that $D \subseteq \mathbf{R}^d$ is open, and every boundary point of D is regular. Let $F \in C_0(D)$. If the distribution $\check{A} * F$ restricted to D is equal to a $C_0(D)$ function, then $F \in \mathcal{D}(\mathcal{A}_D; C_0)$ and $\mathcal{A}_D F(x) = \check{A} * F(x)$ for $x \in D$.*

In the proof, we use the *Dynkin's formula* ([25], formula (5.8)): when τ is a Markov time, $\mathbf{E}_x \tau < \infty$, and $f \in \mathcal{D}(\mathcal{A}; C_0)$, then

$$(2.13) \quad \mathbf{E}_x f(X_\tau) - f(x) = \mathbf{E}_x \left(\int_0^\tau \mathcal{A}f(X_s) ds \right), \quad x \in \mathbf{R}^d.$$

We denote by $B(x, \varepsilon)$ and $\overline{B}(x, \varepsilon)$ the open and closed balls centered at x and with radius ε .

Proof of Lemma 2.10. Fix $x \in D$ and let $r > 0$. Let $g_r \in \mathcal{S}$ be an approximation to identity: $g_r(y) \geq 0$, $\int_{\mathbf{R}^d} g_r(y) dy = 1$ and $g_r(y) = 0$ when $|y| \geq r$. We define $f = F * g_r$. Then $f \in C_b^\infty(\mathbf{R}^d) \subseteq \mathcal{D}(\mathcal{A}; C_b)$ (see Lemma 2.2), and

$$(2.14) \quad \mathcal{A}f(y) = \check{A} * (F * g_r) = (\check{A} * F) * g_r = F * (\check{A} * g_r).$$

(The convolution of \check{A} , F and g_r is associative because F is a bounded function, g_r has compact support, and $\check{A} * \varphi$ is integrable for all $\varphi \in \mathcal{S}$.) In particular, $\mathcal{A}f \in C_0(\mathbf{R}^d)$, and therefore $f \in \mathcal{D}(\mathcal{A}; C_0)$. Let $\tau_{x, \varepsilon}$ be the time of first exit from $B(x, \varepsilon)$. By Dynkin's formula (2.13),

$$\begin{aligned} \left| \frac{\mathbf{E}_x f(X(\tau_{x, \varepsilon})) - f(x)}{\mathbf{E}_x \tau_{x, \varepsilon}} - \mathcal{A}f(x) \right| &= \frac{1}{\mathbf{E}_x \tau_{x, \varepsilon}} \left| \mathbf{E}_x \int_0^{\tau_{x, \varepsilon}} (\mathcal{A}f(X_t) - \mathcal{A}f(x)) dt \right| \\ &\leq \sup_{y \in B(x, \varepsilon)} |\mathcal{A}f(y) - \mathcal{A}f(x)|. \end{aligned}$$

Suppose that $r + \varepsilon < \text{dist}(x, \mathbf{R}^d \setminus D)$. Then using (2.14) for the right-hand side, we obtain

$$\left| \frac{\mathbf{E}_x f(X(\tau_{x, \varepsilon})) - f(x)}{\mathbf{E}_x \tau_{x, \varepsilon}} - \mathcal{A}f(x) \right| \leq \sup_{y \in B(x, r + \varepsilon)} |\check{A} * F(y) - \check{A} * F(x)|.$$

Consider the limit as $r \rightarrow 0^+$. Then g_r approximates the Dirac delta measure δ_0 . Hence, $f = F * g_r$ converges uniformly to F , and (again by (2.14)) $\mathcal{A}f(x)$ converges to $\check{A} * F(x)$ ($x \in D$ is fixed). It follows that

$$\left| \frac{\mathbf{E}_x F(X(\tau_{x, \varepsilon})) - F(x)}{\mathbf{E}_x \tau_{x, \varepsilon}} - \check{A} * F(x) \right| \leq \sup_{y \in \overline{B}(x, \varepsilon)} |\check{A} * F(y) - \check{A} * F(x)|.$$

As $\varepsilon \rightarrow 0^+$, the right-hand side tends to zero. Therefore, $F \in \mathcal{D}(\mathfrak{A}_x)$ and $\mathfrak{A}_x F = \check{A} * F(x)$. Since $x \in D$ was arbitrary, we conclude that $F \in \mathcal{D}(\mathfrak{A}_D)$, and $\mathfrak{A}_D F(x) = \check{A} * F(x)$ for $x \in D$. The result follows from Lemma 2.9. \square

Lemma 2.10 seems to be new. However, a related distributional approach to generators of the free and killed processes was used for example in [8], in the study of harmonic functions with respect to symmetric α -stable processes.

We need the following auxiliary result. Its L^2 analogue is relatively easy to prove using Dirichlet forms. The uniform version seems to be new, especially in the case of unbounded D and F not vanishing at infinity.

Lemma 2.11. *Suppose that $r > 0$, $D \subseteq \mathbf{R}^d$ is an open set, $F \in C_b^\infty(\mathbf{R}^d)$, and $F(x) = 0$ whenever $\text{dist}(x, \mathbf{R}^d \setminus D) \leq r$. Suppose furthermore, that $\mathcal{A}F(x) = 0$ for all $x \in \partial D$. Then $F \in \mathcal{D}(\mathcal{A}_D; C_b)$ and $\mathcal{A}_D F(x) = \mathcal{A}F(x)$ for $x \in D$.*

Recall that $B(x, \varepsilon)$ is the open ball centered at x and with radius ε . In the proof, the following well-known result is used. For completeness, we give a simple proof.

Proposition 2.12. *For all $\varepsilon > 0$, $\mathbf{P}_0(t \geq \tau_{B(0, \varepsilon)})/t$ is bounded in $t > 0$.*

Proof. Let $h \in \mathcal{S}$ satisfy $h(0) = 1$, $h(x) = 0$ for $x \notin B(0, \varepsilon)$, and $h(x) \leq 1$ for all x . Define $\tau = \min(t, \tau_{B(0, \varepsilon)})$. Then, by Dynkin's formula (2.13),

$$\begin{aligned} \mathbf{P}_0(t \geq \tau_{B(0, \varepsilon)}) &= 1 - \mathbf{P}_0(t < \tau_{B(0, \varepsilon)}) \leq h(0) - \mathbf{E}_0 h(X_\tau) \\ &= -\mathbf{E}_x \left(\int_0^\tau \mathcal{A}h(X_s) ds \right) \leq t \|\mathcal{A}h\|_{L^\infty(\mathbf{R}^d)}. \end{aligned} \quad \square$$

Proof of Lemma 2.11. Since there are few tools to handle functions non-vanishing at infinity, we use the definition of \mathcal{A}_D . For $\varepsilon \in (0, r]$, denote by D_ε the set of x such that $\text{dist}(x, \mathbf{R}^d \setminus D) \geq \varepsilon$, and $D'_\varepsilon = \mathbf{R}^d \setminus D_\varepsilon$. Our strategy is to show that away from the boundary of D (in D_ε) we have $P_t^D F \approx P_t f$, while near the boundary (in $D \setminus D_\varepsilon$), $P_t^D F \approx 0$.

Let $g_\varepsilon(t) = \mathbf{P}_x(t \geq \tau_{B(x, \varepsilon)})$. Note that $g_\varepsilon(t)$ does not depend on $x \in \mathbf{R}^d$, and that by Proposition 2.12, $g_\varepsilon(t)/t$ is a bounded function of $t > 0$.

First, we consider $x \in D_\varepsilon$, where $\varepsilon \in (0, r]$ will be specified later. By Lemma 2.2, $F \in \mathcal{D}(\mathcal{A}; C_b)$. Since F vanishes on D'_r , for $x \in D_\varepsilon$ we have

$$\begin{aligned} |P_t F(x) - P_t^D F(x)| &= |\mathbf{E}_x(F(X_t) \mathbf{1}_{t \geq \tau_D})| \\ &\leq \|F\|_{L^\infty(\mathbf{R})} \mathbf{E}_x(\mathbf{1}_{X_t \in D_r} \mathbf{1}_{t \geq \tau_D}). \end{aligned}$$

Observe that the condition $X_t \in D_r$ and $t \geq \tau_D$ implies that before time t , the process *first* exits D , and *after that* it exits D'_r . Furthermore, $B(X_{\tau_D}, \varepsilon) \subseteq D'_r$ a.s. Hence, by the strong Markov property,

$$\begin{aligned} \mathbf{E}_x(\mathbf{1}_{X_t \in D_r} \mathbf{1}_{t \geq \tau_D}) &\leq \mathbf{P}_x(X_s \text{ first exits } D \text{ and then } B(X_{\tau_D}, \varepsilon) \text{ before time } t) \\ &= \mathbf{E}_x(g_\varepsilon(t - \tau_D) \mathbf{1}_{t \geq \tau_D}). \end{aligned}$$

Since g_ε is increasing, and $B(x, \varepsilon) \subseteq D$, we have

$$\begin{aligned} \mathbf{E}_x(\mathbf{1}_{X_t \in D} \mathbf{1}_{t \geq \tau_D}) &\leq g_\varepsilon(t) \mathbf{P}_x(t \geq \tau_D) \\ &\leq g_\varepsilon(t) \mathbf{P}_x(t \geq \tau_{B(x, \varepsilon)}) = (g_\varepsilon(t))^2. \end{aligned}$$

We conclude that for $x \in D_\varepsilon$,

$$(2.15) \quad \left| \frac{P_t^D F(x) - F(x)}{t} - \mathcal{A}F(x) \right| \leq \left\| \frac{P_t F - F}{t} - \mathcal{A}F \right\|_{L^\infty(\mathbf{R}^d)} + \frac{\|F\|_{L^\infty(\mathbf{R})} (g_\varepsilon(t))^2}{t}.$$

Now we consider x close to the boundary. Fix (small) $\eta > 0$. By Corollary 2.6, $\mathcal{A}F$ is Lipschitz continuous. Hence, we can choose $\varepsilon \in (0, r/2]$ small enough, so that $|\mathcal{A}F(y)| \leq \eta$ when $\text{dist}(y, D \setminus D_\varepsilon) < \varepsilon$. Let $\varphi_n \in \mathcal{S}$ be the sequence approximating F as in the proof of Lemma 2.2. Let $x \in D \setminus D_\varepsilon$ and $\tau = \min(t, \tau_D)$. By Dynkin's formula (2.13),

$$|\mathbf{E}_x \varphi_n(X_\tau) - \varphi_n(x)| = \left| \mathbf{E}_x \left(\int_0^\tau \mathcal{A} \varphi_n(X_s) ds \right) \right| \leq \mathbf{E}_x \left(\int_0^t |\mathcal{A} \varphi_n(X_s)| ds \right).$$

When $t < \tau_{B(x, \varepsilon)}$, we use the estimate

$$\int_0^t |\mathcal{A} \varphi_n(X_s)| ds \leq t \sup_{y \in B(x, \varepsilon)} |\mathcal{A} \varphi_n(y)|.$$

When $t \geq \tau_{B(x, \varepsilon)}$, we simply have

$$\int_0^t |\mathcal{A} \varphi_n(X_s)| ds \leq t \|\mathcal{A} \varphi_n\|_{L^\infty(\mathbf{R}^d)}.$$

It follows that

$$\begin{aligned} |\mathbf{E}_x \varphi_n(X_\tau) - \varphi_n(x)| &\leq t \mathbf{P}_x(t < \tau_{B(x, \varepsilon)}) \sup_{y \in B(x, \varepsilon)} |\mathcal{A} \varphi_n(y)| \\ &\quad + t \mathbf{P}_x(t \geq \tau_{B(x, \varepsilon)}) \|\mathcal{A} \varphi_n\|_{L^\infty(\mathbf{R}^d)} \\ &\leq t \left(\sup_{y \in B(x, \varepsilon)} |\mathcal{A} \varphi_n(y)| + g_\varepsilon(t) \|\mathcal{A} \varphi_n\|_{L^\infty(\mathbf{R}^d)} \right). \end{aligned}$$

Recall that φ_n and $\mathcal{A} \varphi_n$ converge to F and $\mathcal{A}F$, respectively, locally uniformly, and that $\|\mathcal{A} \varphi_n\|_{C_b^2(\mathbf{R}^d)} \leq C \|\varphi_n\|_{C_b^4(\mathbf{R}^d)} \leq C^2 \|F\|_{C_b^4(\mathbf{R}^d)}$ for some $C > 0$. By taking the limit as $n \rightarrow \infty$ and applying dominated convergence, we obtain that for $x \in D \setminus D_\varepsilon$,

$$|\mathbf{E}_x F(X_\tau) - F(x)| \leq t \left(\sup_{y \in B(x, \varepsilon)} |\mathcal{A}F(y)| + C^2 g_\varepsilon(t) \|F\|_{C_b^4(\mathbf{R}^d)} \right).$$

We have $|\mathcal{A}F(y)| \leq \eta$ for $y \in B(x, \varepsilon)$ (because $\text{dist}(y, D \setminus D_\varepsilon) < \varepsilon$). Hence,

$$\begin{aligned} \left| \frac{\mathbf{E}_x F(X_\tau) - F(x)}{t} - \mathcal{A}F(x) \right| &\leq \frac{|\mathbf{E}_x F(X_\tau) - F(x)|}{t} + |\mathcal{A}F(x)| \\ &\leq 2\eta + C^2 g_\varepsilon(t) \|F\|_{C_b^4(\mathbf{R}^d)}. \end{aligned}$$

Finally, $F(X_{\tau_D}) = 0$ a.s., and therefore

$$\mathbf{E}_x F(X_\tau) = \mathbf{E}_x (F(X_t) \mathbf{1}_{t < \tau_D}) = P_t^D F(x).$$

We conclude that

$$(2.16) \quad \left| \frac{P_t^D F(x) - F(x)}{t} - \mathcal{A}F(x) \right| \leq 2\eta + Cg_\varepsilon(t) \|F\|_{C_b^4(\mathbf{R}^d)}.$$

Recall that as $t \rightarrow 0^+$, $g_\varepsilon(t)/t$ is bounded. Furthermore, $(P_t F - F)/t$ converges uniformly to $\mathcal{A}F$. Hence, (2.15) and (2.16) imply that for t small enough,

$$\left\| \frac{P_t^D F - F}{t} - \mathbf{1}_D \mathcal{A}F \right\|_{L^\infty(D)} \leq 3\eta.$$

Since $\eta > 0$ was arbitrary, the proof is complete. \square

2.4. Subordinate Brownian motions. From now on, we only consider one-dimensional Lévy processes. The process X_t corresponding to $\beta = 1$, $\gamma = 0$ and ν vanishing, is the Brownian motion, running at twice the usual speed (that is, $\text{Var } X_t = 2t$). We denote the transition density of the Brownian motion by $k_s(z)$,

$$(2.17) \quad k_s(z) = \frac{1}{\sqrt{4\pi s}} \exp\left(-\frac{z^2}{4s}\right), \quad s > 0, z \in \mathbf{R}.$$

A Lévy process X_t is said to be a *subordinate Brownian motion* if it can be written in the form $X_t = B_{Z_t}$, where B_s is the Brownian motion, Z_t is a *subordinator* (a non-decreasing Lévy process), and B_s and Z_t are independent processes. We assume that under \mathbf{P}_x , B_s starts at x and Z_t starts at 0 a.s. To avoid trivialities, we assume that X_t is non-constant, that is, $\beta > 0$ or ν is non-zero.

The subordinator Z_t is completely described by its *Laplace exponent* $\psi(\xi)$: we have $\mathbf{E}_0 e^{-\xi Z_t} = e^{-t\psi(\xi)}$ ($\xi \in \mathbf{C}$, $\text{Re } \xi > 0$), where

$$(2.18) \quad \psi(\xi) = \beta\xi + \int_0^\infty (1 - e^{-\xi s})\mu(ds), \quad \text{Re } \xi > 0.$$

Here $\beta \geq 0$ is the same as in (2.10), and μ is the Lévy measure of Z_t : a Radon measure on $(0, \infty)$ satisfying $\int_0^\infty \min(1, s)\mu(ds) < \infty$. The relation between the Lévy measures μ and ν of Z_t and X_t is

$$(2.19) \quad \nu(z) = \int_0^\infty k_s(z)\mu(ds), \quad z \in \mathbf{R} \setminus \{0\},$$

with $k_s(z)$ defined in (2.17). The Lévy-Khintchine exponent of the subordinate Brownian motion X_t satisfies $\Psi(\xi) = \psi(\xi^2)$.

The class of subordinate Brownian motions is naturally divided into compound Poisson processes (with bounded Laplace exponents ψ) and processes corresponding to unbounded ψ . The latter class consists of strong Feller processes: if $Z_t > 0$ a.s., then $X_t = B_{S_t}$ has absolutely continuous distribution. This property is important for applications of Lemma 2.10. See [10, 44, 66] for more information on subordinate Brownian motions.

Remark 2.13. One needs to distinguish the *killed subordinate* and the *subordinate killed* processes. The first one is X_t^D introduced above, while the other one is obtained by subordination of the killed Brownian motion. We emphasize that these two processes are essentially different. In particular, the spectral theory for the subordinate killed process is trivial: its eigenfunctions are the same as the eigenfunctions of the (insubordinate) killed Brownian motion, and only the corresponding eigenvalues are different. For a discussion of the relation between killed subordinate and subordinate killed processes, see e.g. [69]. \square

We recall some standard definitions; see [66] for further information. A function $f : (0, \infty) \rightarrow \mathbf{R}$ is said to be *completely monotone* on $(0, \infty)$ if $f^{(n)}(z) \geq 0$ for all $z > 0$ and $n = 0, 1, 2, \dots$. A function $f : (0, \infty) \rightarrow \mathbf{R}$ is a *Bernstein function* if f is nonnegative and f' is completely monotone. By Bernstein's theorem, every completely monotone function is the Laplace transform of some Radon measure on $[0, \infty)$, and every Bernstein function has the form

$$(2.20) \quad f(z) = c_1 + c_2 z + \int_0^\infty (1 - e^{-zs}) m(ds), \quad z > 0,$$

for some $c_1, c_2 \geq 0$ and a Radon measure m on $(0, \infty)$ satisfying $\int_0^\infty \min(1, s) m(ds) < \infty$. Whenever we write $f(0)$ when f is a Bernstein function, we mean $\lim_{z \rightarrow 0^+} f(z)$. Note that if $f(0) = 0$, then f has the form (2.18); hence, a Bernstein function f is the Laplace exponent of a subordinator if and only if $f(0) = 0$.

2.5. Complete Bernstein functions and Stieltjes functions. A Bernstein function f for which the measure m in the representation (2.20) has a completely monotone density function is said to be a *complete Bernstein function* (CBF in short; also known as *operator monotone function*). Recall that X_t is a subordinate Brownian motion, μ is the Lévy measure of the underlying subordinator, and ψ is its Laplace exponent (so that the Lévy-Khintchine exponent of X_t is $\psi(\xi^2)$).

Proposition 2.14. *The following conditions are equivalent:*

- (a) ψ is a complete Bernstein function;
- (b) $\mu(ds)$ has a completely monotone density function $\mu(s)$;
- (c) for a Radon measure μ_0 on $(0, \infty)$ such that the integral $\int_0^\infty \min(\zeta^{-1}, \zeta^{-2}) \mu_0(d\zeta)$ is finite,

$$(2.21) \quad \psi(\xi) = \beta \xi + \frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi + \zeta} \frac{\mu_0(d\zeta)}{\zeta};$$

- (d) ψ extends to a holomorphic function in $\mathbf{C} \setminus (-\infty, 0]$, which leaves the upper and the lower complex half-planes invariant;
- (e) $\nu(z)$ is a completely monotone function on $(0, \infty)$.

A symmetric Lévy process satisfying (e) is automatically a subordinate Brownian motion.

Proof. Equivalence of (a) and (b) is just the definition of complete Bernstein functions. Equivalence of (b)–(d) is standard, see [66], Chapters 6 and 11. Equivalence of (c) and (e) seems to be well-known to specialists (cf. [62]), but a direct reference to the literature is difficult to find. The proof is a rather straightforward calculation involving Fourier transform and an application of Bernstein’s representation theorem; for completeness, we provide the details.

Suppose that (c) holds. Then, for $\xi \in \mathbf{R}$,

$$\Psi(\xi) = \psi(\xi^2) = \beta\xi^2 + \frac{1}{\pi} \int_0^\infty \frac{\xi^2}{\xi^2 + \zeta} \frac{\mu_0(d\zeta)}{\zeta}.$$

By an explicit calculation, for $\zeta > 0$,

$$\int_{-\infty}^\infty (1 - \cos(\xi z)) e^{-\sqrt{\zeta}|z|} dz = \frac{2}{\sqrt{\zeta}} - \frac{2\sqrt{\zeta}}{\xi^2 + \zeta} = \frac{2}{\sqrt{\zeta}} \frac{\xi^2}{\xi^2 + \zeta}.$$

Hence, by Fubini,

$$\begin{aligned} \Psi(\xi) &= \beta\xi^2 + \frac{1}{2\pi} \int_0^\infty \left(\int_{-\infty}^\infty (1 - \cos(\xi z)) e^{-\sqrt{\zeta}|z|} dz \right) \frac{\mu_0(d\zeta)}{\sqrt{\zeta}} \\ &= \beta\xi^2 + \int_{-\infty}^\infty (1 - \cos(\xi z)) \left(\frac{1}{2\pi} \int_0^\infty e^{-\sqrt{\zeta}|z|} \frac{\mu_0(d\zeta)}{\sqrt{\zeta}} \right) dz. \end{aligned}$$

Comparing this with (2.10), we obtain

$$(2.22) \quad \nu(z) = \frac{1}{2\pi} \int_0^\infty e^{-\sqrt{\zeta}|z|} \frac{\mu_0(d\zeta)}{\sqrt{\zeta}},$$

which proves that $\nu(z)$ is completely monotone on $(0, \infty)$. Conversely, if (e) holds, that is, if $\nu(z)$ is completely monotone on $(0, \infty)$, then, by Bernstein’s theorem, $\nu(z)$ has the representation (2.22), and the above reasoning can be reversed to prove that X_t is a subordinate Brownian motion, and that (c) holds. \square

The assumptions of the main theorems can be put in the following form.

Assumption 2.15. The process X_t is a subordinate Brownian motion satisfying any of the equivalent conditions of Proposition 2.14.

We note the following immediate consequence of Proposition 2.14.

Corollary 2.16. *The following conditions are equivalent:*

- (a) *f is a complete Bernstein function; that is, (2.20) holds for some $c_1, c_2 \geq 0$ and a measure m with completely monotone density;*
- (b) *for some $c_1, c_2 \geq 0$ and a Radon measure m_0 on $(0, \infty)$ such that $\int_0^\infty \min(s^{-1}, s^{-2}) m_0(ds) < \infty$,*

$$(2.23) \quad f(z) = c_1 + c_2 z + \frac{1}{\pi} \int_0^\infty \frac{z}{z + s} \frac{m_0(ds)}{s}, \quad z > 0;$$

- (c) *$f(z) \geq 0$ for $z > 0$ and either f is constant, or f extends to a holomorphic function in $\mathbf{C} \setminus (-\infty, 0]$, which leaves the upper and the lower complex half-planes invariant;*

- (d) $f(\xi^2) - f(0)$ is the Lévy-Khintchine exponent of a symmetric Lévy process, whose Lévy measure has a completely monotone density function on $(0, \infty)$. \square

The right-hand side of formula (2.23) defines a holomorphic function in $\mathbf{C} \setminus (-\infty, 0]$. We often denote this extension by the same symbol f . We will also use the related notion of a Stieltjes function: f is said to be a *Stieltjes function* if there are $\tilde{c}_1, \tilde{c}_2 \geq 0$ and a Radon measure \tilde{m}_0 on $(0, \infty)$ satisfying $\int_0^\infty \min(1, s^{-1}) \tilde{m}_0(ds) < \infty$ such that

$$(2.24) \quad f(z) = \tilde{c}_1 + \frac{\tilde{c}_2}{z} + \frac{1}{\pi} \int_0^\infty \frac{1}{z+s} \tilde{m}_0(ds), \quad z > 0.$$

Since $1/(z+s) = \int_0^\infty e^{-\xi s} e^{-\xi z} d\xi$, we see that f has the representation (2.24) if and only if it is the Laplace transform of the measure

$$\tilde{c}_1 \delta_0(d\xi) + (\tilde{c}_2 + \pi^{-1} \mathcal{L} \tilde{m}_0(\xi)) d\xi$$

on $[0, \infty)$, where δ_0 is the Dirac delta measure. In other words, f is the Laplace transform of a completely monotone function, plus a nonnegative constant.

We list some standard properties of complete Bernstein and Stieltjes functions.

Proposition 2.17 (Chapter 3 in [37] and Chapter 7 in [66]). *Suppose that f and g are not constantly equal to 0.*

- (a) *The following conditions are equivalent: $f(z)$ is a CBF; $z/f(z)$ is a CBF; $1/f(z)$ is a Stieltjes function; $f(z)/z$ is a Stieltjes function.*
- (b) *If f, g are CBF, $a, C > 0$ and $\alpha \in (0, 1)$, then $f(z) + g(z)$, $Cf(z)$, $f(Cz)$, $(z-a)/(f(z)-f(a))$ (extended continuously at $z=a$), $f(g(z))$ and $(f(z^\alpha))^{1/\alpha}$ are CBF.* \square

One of the fundamental properties of complete Bernstein and Stieltjes functions is that the characteristics in representations (2.23) and (2.24) can be easily recovered from (the holomorphic extension of) the function. We illustrate this for Stieltjes functions. By δ_0 we denote the Dirac delta measure at 0.

Proposition 2.18. *Let f be (the holomorphic extension of) a Stieltjes function with representation (2.24). Then*

$$(2.25) \quad \tilde{c}_1 = \lim_{z \rightarrow \infty} f(z), \quad \tilde{c}_2 = \lim_{z \rightarrow 0^+} (zf(z)),$$

and

$$(2.26) \quad \tilde{m}_0(ds) + \pi \tilde{c}_2 \delta_0(ds) = \lim_{\varepsilon \rightarrow 0^+} (-\operatorname{Im} f(-s + i\varepsilon) ds),$$

with the limit understood in the sense of weak convergence of measures.

Formula (2.26) is well-known in complex analysis, it is a variant of the Sokhotskyi-Plemelj formula for the Cauchy-Stieltjes transform in the upper complex half-plane. The proof of a version for complete Bernstein functions is contained in Theorem 6.2 in [66] (see Corollary 6.3 and Remark 6.4

therein). We provide a simple argument based on the representation theorem for harmonic functions in half-plane.

Proof of Proposition 2.18. Since $-\operatorname{Im} f$ is a nonnegative harmonic function in the upper complex half-plane, it can be represented using the Poisson kernel,

$$(2.27) \quad -\operatorname{Im} f(x + iy) = Cy + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x - s)^2} \tilde{m}(ds)$$

for unique constant $C = \lim_{y \rightarrow \infty} (-\operatorname{Im} f(iy)/y)$ and unique measure $\tilde{m}(ds) = \lim_{\varepsilon \rightarrow 0^+} (-\operatorname{Im} f(s + i\varepsilon)ds)$. On the other hand, by (2.24),

$$-\operatorname{Im} f(x + iy) = \frac{\tilde{c}_2 y}{y^2 + x^2} + \frac{1}{\pi} \int_0^{\infty} \frac{y}{y^2 + (x + s)^2} \tilde{m}_0(ds).$$

Formula (2.26) follows simply by comparing the above two representations. The expressions (2.25) are direct consequences of (2.24) and dominated convergence. \square

We need three more properties of complete Bernstein functions.

Proposition 2.19. *If f is a complete Bernstein function and $a > 0$, then $g(z) = (1 - z/a)/(1 - f(z)/f(a))$ and $h(z) = \log g(z)$ are complete Bernstein functions.*

Here we extend g and h continuously at $z = a$, $g(a) = f(a)/(af'(a))$ and $h(a) = \log(f(a)/(af'(a)))$.

Proof. By Proposition 2.17(b), $g(z) = (f(a)/a)(x - a)/(f(z) - f(a))$ is a CBF. Note that $g(0) = 1$, and so also $g(z) - 1$ is a CBF. Furthermore, $\log(1 + z)$ is a CBF. Again by Proposition 2.17(b), $h(z) = \log(1 + (g(z) - 1))$ is a CBF. \square

Lemma 2.20. *Let f be (the holomorphic extension of) a complete Bernstein function, and let $a > 0$. Define*

$$f^*(z) = \frac{1}{2ia} \left(\frac{f(ia)}{z - ia} - \frac{f(-ia)}{z + ia} \right), \quad z \in \mathbf{C} \setminus \{-ia, ia\}.$$

Then $g(z) = f^(z) - f(z)/(z^2 + a^2)$ is a Stieltjes function, and it is the Laplace transform of a completely monotone function G on $(0, \infty)$. Furthermore, G is the Laplace transform of a finite measure on $(0, \infty)$.*

Proof. Let f have the representation (2.23), and let $h(z) = f(z) - c_1 - c_2 z$. Fix $z \in \mathbf{C} \setminus (-\infty, 0]$. Since

$$\begin{aligned} \frac{1}{z + s} \frac{1}{a^2 + s^2} &= \left(\frac{1}{2ia(z - ia)} \frac{ia}{s + ia} \right. \\ &\quad \left. + \frac{1}{2(-ia)(z + ia)} \frac{-ia}{s - ia} - \frac{1}{a^2 + z^2} \frac{z}{z + s} \right) \frac{1}{s}, \end{aligned}$$

we have

$$\frac{1}{\pi} \int_0^{\infty} \frac{1}{z + s} \frac{1}{a^2 + s^2} m_0(ds) = \frac{1}{2ia} \left(\frac{h(ia)}{z - ia} - \frac{h(-ia)}{z + ia} \right) - \frac{h(z)}{a^2 + z^2}.$$

Furthermore,

$$\frac{1}{2ia} \left(\frac{c_1 + c_2 ia}{z - ia} - \frac{c_1 - c_2 ia}{z + ia} \right) - \frac{c_1 + c_2 z}{a^2 + z^2} = 0.$$

We conclude that

$$\frac{1}{\pi} \int_0^\infty \frac{1}{z + s} \frac{m_0(ds)}{a^2 + s^2} = \frac{1}{2ia} \left(\frac{f(ia)}{z - ia} - \frac{f(-ia)}{z + ia} \right) - \frac{f(z)}{a^2 + z^2} = g(z).$$

By the definition, g is the Stieltjes function, and by the remark following (2.24), g is the Laplace transform of a completely monotone function $G(x) = \mathcal{L}\gamma(x)$, where $\gamma(ds) = (1/\pi)(a^2 + s^2)^{-1}m_0(ds)$ is a finite measure on $(0, \infty)$. \square

Proposition 2.21. *If f is a complete Bernstein function, then*

- (a) $0 \leq zf'(z) \leq f(z)$ for $z > 0$;
- (b) $0 \leq -zf''(z) \leq 2f'(z)$ for $z > 0$;
- (c) $|f(z)| \leq (\sin(\varepsilon/2))^{-1} f(|z|)$ for $z \in \mathbf{C}$, $|\operatorname{Arg} z| \leq \pi - \varepsilon$, $\varepsilon \in (0, \pi)$;
- (d) $|zf'(z)| \leq (\sin(\varepsilon/2))^{-1} f(|z|)$ for z as in (c);
- (e) $|f(z)| \leq C(f, \varepsilon)(1 + |z|)$ for z as in (c).

In (c)–(e), the holomorphic extension of f is denoted by the same symbol.

Proof. All statements follow easily from the representation (2.23) and differentiation under the integral sign. Indeed, (a) reduces to

$$0 \leq c_2 z + \frac{1}{\pi} \int_0^\infty \frac{sz}{(s+z)^2} \frac{m_0(ds)}{s} \leq c_1 + c_2 z + \frac{1}{\pi} \int_0^\infty \frac{z}{s+z} \frac{m_0(ds)}{s},$$

which follows from the inequality $sz/(s+z)^2 \leq z/(s+z)$ for $s, z > 0$. In a similar manner, (b) follows from $2sz/(s+z)^3 \leq 2s/(s+z)^2$. To prove (c)–(e), observe that for $z \in \mathbf{C}$ with $\operatorname{Arg} z \in [-\pi + \varepsilon, \pi - \varepsilon]$,

$$|s+z|^2 = s^2 + 2s \operatorname{Re} z + |z|^2 \geq s^2 - 2s|z| \cos(\varepsilon) + |z|^2.$$

Hence, by a simple calculation,

$$|s+z|^2 \geq \frac{1 - \cos(\varepsilon)}{2} (s + |z|)^2 = (\sin(\varepsilon/2))^2 (s + |z|)^2.$$

Therefore, (c) is a consequence of $|z/(s+z)| \leq (\sin(\varepsilon/2))^{-1} |z|/(s+|z|)$, and (d) is proved using $|sz/(s+z)^2| \leq |z|/|s+z| \leq (\sin(\varepsilon/2))^{-1} |z|/(s+|z|)$. Finally, to prove (e), use (c) and the inequality $|z|/(s+|z|) \leq (1+|z|)/(s+1)$, verified easily by a direct calculation. \square

For further properties of complete Bernstein functions and related notions, see Chapters 6 and 7 in [66], or Chapter 3 in [37].

3. TRANSFORMATION OF THE LAPLACE EXPONENT

Recall that $X_t = B_{Z_t}$ is a subordinate Brownian motion, $\psi(\xi)$ is the Laplace exponent of the subordinator Z_t , and $\Psi(\xi) = \psi(\xi^2)$ is the Lévy-Khintchine exponent of X_t . Assumption 2.15 is in force; that is, ψ is assumed to be a *complete* Bernstein function. In this section we introduce the operation $\psi \mapsto \psi^\dagger$. Although it will be used for complete Bernstein functions only, the definition is quite general.

Definition 3.1. When ψ is a positive function on $(0, \infty)$, and the integral $\int_0^\infty \min(1, \xi^{-2}) \log \psi(\xi) d\xi$ is finite, then we define

$$(3.1) \quad \psi^\dagger(\xi) = \exp \left(\frac{1}{\pi} \int_0^\infty \frac{\log \psi(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta \right), \quad \xi > 0.$$

Proposition 3.2. For ψ as Definition 3.1,

$$(3.2) \quad \psi^\dagger(\xi) = \exp \left(\frac{1}{\pi} \int_0^\infty \frac{\xi \log \psi(\zeta^2)}{\xi^2 + \zeta^2} d\zeta \right), \quad \xi > 0.$$

Proof. Substitute $\xi\zeta = s$ in (3.1). \square

Note that (3.2) defines a holomorphic extension of ψ^\dagger to the half-plane $\operatorname{Re} \xi > 0$. We denote this extension by the same symbol $\psi^\dagger(\xi)$.

Remark 3.3. The definition of ψ^\dagger lies at the very heart of the Wiener-Hopf method. Roughly speaking, if $\psi^\dagger(\xi)$ can be extended continuously to the boundary of the region $\operatorname{Re} \xi > 0$, that is, to the imaginary axis, then we have $\psi(\xi^2) = \psi^\dagger(i\xi)\psi^\dagger(-i\xi)$, a Wiener-Hopf factorization of $\psi(\xi^2)$. For complete Bernstein functions, this is formally proved in Lemma 3.8 below. The Wiener-Hopf method is described in Section 4.

The function ψ^\dagger plays an important role in fluctuation theory of Lévy processes. When X_t is a symmetric Lévy process, then its Lévy-Khintchine exponent Ψ is symmetric, $\Psi(\xi) = \psi(\xi^2)$ for some nonnegative ψ on $[0, \infty)$. Suppose that X_t is not a compound Poisson process. By Corollary 9.7 in [28], the Laplace exponent $\kappa(z, \xi)$ of the bivariate ascending ladder process satisfies

$$\kappa(z, \xi) = (z + \psi)^\dagger(\xi), \quad z \geq 0, \xi > 0.$$

This relation was used extensively in [43, 44, 55]. \square

The fundamental Lemma 3.8 states that if ψ is a CBF, then ψ^\dagger is a CBF. Before we prove it, we establish some simple properties of ψ^\dagger .

Proposition 3.4. Whenever both sides of the following identities make sense, we have:

- (a) $(1/\psi)^\dagger = 1/\psi^\dagger$, $(\psi^\alpha)^\dagger = (\psi^\dagger)^\alpha$ ($\alpha \in \mathbf{R}$) and $(\psi_1\psi_2)^\dagger = \psi_1^\dagger\psi_2^\dagger$;
- (b) $(C^2\psi)^\dagger = C\psi^\dagger$ for $C > 0$;
- (c) when $\psi(\xi) = \xi$, then $\psi^\dagger(\xi) = \xi$;
- (d) if appropriate limits of ψ exist, then the corresponding limits of ψ^\dagger exist, and $\lim_{\xi \rightarrow 0+} \psi^\dagger(\xi) = (\lim_{\xi \rightarrow 0+} \psi(\xi))^{1/2}$, $\lim_{\xi \rightarrow \infty} \psi^\dagger(\xi) = (\lim_{\xi \rightarrow \infty} \psi(\xi))^{1/2}$.

Proof. Properties (a) follow immediately from formula (3.2). In (b), one uses also $\int_0^\infty \xi/(\xi^2 + \zeta^2) d\zeta = \pi/2$. For $\psi(\xi) = \xi$, by (3.1) we have

$$\begin{aligned} \psi^\dagger(\xi) &= \exp \left(\frac{1}{\pi} \int_0^\infty \frac{2 \log \xi + 2 \log \zeta}{1 + \zeta^2} d\zeta \right) \\ &= \exp \left(\log \xi + \frac{2}{\pi} \int_0^\infty \frac{\log \zeta}{1 + \zeta^2} d\zeta \right) = \xi; \end{aligned}$$

here we used the identity $\int_0^\infty (1+s^2)^{-1} \log s \, ds = 0$, proved easily by a substitution $r = 1/s$. Finally, to show (d), assume that a finite, positive limit $\psi(0) = \lim_{\xi \rightarrow 0^+} \psi(\xi)$ exists. It can be proved that the functions $\log \psi(\xi^2 \zeta^2)/(1+\zeta^2)$ of ζ are uniformly integrable as $\xi \rightarrow 0^+$ (this is clear when ψ is a CBF; we omit the details in the general case). It follows that

$$\lim_{\xi \rightarrow 0^+} \int_0^\infty \frac{\log \psi(\xi^2 \zeta^2)}{1+\zeta^2} d\zeta = \int_0^\infty \frac{\log \psi(0)}{1+\zeta^2} d\zeta = \frac{\pi \log \psi(0)}{2},$$

and hence $\lim_{\xi \rightarrow 0^+} \psi^\dagger(\xi) = \sqrt{\psi(0)}$. When the limit of ψ is 0 or ∞ , one uses in a similar way Fatou's lemma; we omit the details. Limits at ∞ are dealt with in a similar way. \square

Proposition 3.5. *Let ψ be as in Definition 3.1. If $C_1, C_2 > 0$, $\alpha \in \mathbf{R}$ and $\psi(\xi) \leq (C_1^2 + C_2^2 \xi)^\alpha$ for all $\xi > 0$, then, with the same constants, $|\psi^\dagger(\xi)| \leq |C_1 + C_2 \xi|^\alpha$, $\operatorname{Re} \xi > 0$. In a similar manner, if $\psi(\xi) \geq (C_1^2 + C_2^2 \xi)^\alpha$ for $\xi > 0$ for some $C_1, C_2 > 0$, $\alpha \in \mathbf{R}$, then $|\psi^\dagger(\xi)| \geq |C_1 + C_2 \xi|^\alpha$, $\operatorname{Re} \xi > 0$.*

Proof. For $\xi = t + is$ ($t > 0$, $s \in \mathbf{R}$), we have

$$\begin{aligned} |\psi^\dagger(\xi)|^2 &= \exp \left(\frac{1}{\pi} \int_0^\infty \operatorname{Re} \frac{2\xi}{\xi^2 + \zeta^2} \log \psi(\zeta^2) d\zeta \right) \\ &= \exp \left(\frac{1}{\pi} \int_0^\infty \operatorname{Re} \left(\frac{1}{\xi - i\zeta} + \frac{1}{\xi + i\zeta} \right) \log \psi(\zeta^2) d\zeta \right) \\ &= \exp \left(\frac{1}{\pi} \int_0^\infty \left(\frac{t}{t^2 + (\zeta - s)^2} + \frac{t}{t^2 + (\zeta + s)^2} \right) \log \psi(\zeta^2) d\zeta \right) \\ &= \exp \left(\frac{1}{\pi} \int_{-\infty}^\infty \frac{t}{t^2 + (\zeta - s)^2} \log \psi(\zeta^2) d\zeta \right). \end{aligned}$$

Suppose that $\psi(\zeta) \leq (C_1^2 + C_2^2 \zeta)^\alpha$ for $\zeta > 0$. Then

$$\log \psi(\zeta^2) \leq \alpha \log(C_1^2 + C_2^2 \zeta^2) = 2\alpha \operatorname{Re}(\log(C_1 - C_2 i \zeta)),$$

and the right-hand side extends to a bounded below harmonic function in the upper half-plane $\operatorname{Im} \zeta > 0$ (with vanishing linear term in the representation similar to (2.27)). By the Poisson formula,

$$\frac{1}{\pi} \int_{-\infty}^\infty \frac{t}{t^2 + (\zeta - s)^2} \operatorname{Re}(\log(C_1 - C_2 i \zeta)) d\zeta = \operatorname{Re}(\log(C_1 - C_2 i(s + it))).$$

Hence,

$$\begin{aligned} |\psi^\dagger(\xi)|^2 &\leq \exp(2\alpha \operatorname{Re}(\log(C_1 - C_2 i(s + it)))) \\ &= ((C_1 + C_2 t)^2 + C_2^2 s^2)^\alpha = |C_1 + C_2 \xi|^{2\alpha} \end{aligned}$$

This proves the first statement. The other one follows by reversing the inequalities in the above argument. \square

Remark 3.6. In Proposition 2.1 in [55] it is proved that if $\psi(\xi)$ and $\xi/\psi(\xi)$ are increasing, then $\psi(\xi)/2 \leq \psi^\dagger(\xi) \leq 2\psi(\xi)$. For complete Bernstein functions, this was independently proved in Proposition 3.7 in [44]. \square

Recall that under Assumption 2.15, ψ is a complete Bernstein function. Hence, ψ extends to a holomorphic function in $\mathbf{C} \setminus (-\infty, 0]$; we use the same symbol ψ for this extension. First, we define an auxiliary function

$$(3.3) \quad f(\xi) = \exp \left(\frac{1}{\pi} \int_0^\infty \frac{\log \psi(\xi \zeta^2)}{1 + \zeta^2} d\zeta \right), \quad \xi \in \mathbf{C} \setminus (-\infty, 0],$$

so that $\psi^\dagger(\xi) = f(\xi^2)$. Note that $(f(\xi))^2$ is an (integral-type) weighted geometric mean of the family of complete Bernstein functions $\psi(\xi \zeta^2)$ ($\zeta > 0$), so that $(f(\xi))^2$ is a complete Bernstein function. This is formally proved in the following simple result.

Lemma 3.7. *If ψ is a complete Bernstein function, then $f(\xi)$ and $(f(\xi))^2$ are complete Bernstein functions.*

Proof. Clearly, $f(\xi) \geq 0$ for $\xi > 0$. Furthermore,

$$\operatorname{Arg} f(\xi) = \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im} \log \psi(\xi \zeta^2)}{1 + \zeta^2} d\zeta = \frac{1}{\pi} \int_0^\infty \frac{\operatorname{Arg} \psi(\xi \zeta^2)}{1 + \zeta^2} d\zeta.$$

When $\operatorname{Im} \xi > 0$, we have $\operatorname{Arg} \psi(\xi \zeta^2) \in (0, \pi)$. It follows that $\operatorname{Arg} f(\xi) \in (0, \pi/2)$, that is, $\operatorname{Im} f(\xi) > 0$ and $\operatorname{Im}(f(\xi))^2 > 0$. Finally, $f(\bar{\xi}) = \overline{f(\xi)}$, so $\operatorname{Im} f(\xi) < 0$ and $\operatorname{Im}(f(\xi))^2 < 0$ when $\operatorname{Im} \xi < 0$. The result follows by Corollary 2.16(c). \square

Below we prove that $\psi^\dagger(\xi) = f(\xi^2)$ is a complete Bernstein function; this is stronger than the assertion of Lemma 3.7, see Proposition 2.17(b).

Lemma 3.8. *If ψ is a complete Bernstein function, then ψ^\dagger is a complete Bernstein function. The holomorphic extension of ψ^\dagger (denoted by the same symbol) satisfies*

$$(3.4) \quad \psi^\dagger(\xi)\psi^\dagger(-\xi) = \psi(-\xi^2), \quad \xi \in \mathbf{C} \setminus \mathbf{R}.$$

Remark 3.9. The first statement of the above lemma was proved independently in Proposition 2.4 in [43] using Theorem 6.10 in [66]. After the preliminary version of the article was made available, Jacek Małecki pointed out to the author that the other statement of the lemma, formula (3.4), at least for purely imaginary ξ , can be deduced from the Wiener-Hopf factorization of the Lévy-Khintchine exponent in fluctuation theory, see formula (VI.4) in [5]. The novelty of Lemma 3.8 lies in the combination of the two parts, which is one of the key steps in the derivation of the explicit formula for the eigenfunctions F_λ in Theorem 1.1. \square

Proof. Let f be (the holomorphic extension of) the complete Bernstein function defined in (3.3) (see Lemma 3.7); hence, $\psi^\dagger(\xi) = f(\xi^2)$ for $\xi > 0$, and this identity defines a holomorphic extension of ψ^\dagger to the half-plane $\operatorname{Re} \xi > 0$. When $\operatorname{Re} \xi > 0$, by (3.3) we have

$$\psi^\dagger(\xi) = f(\xi^2) = \exp \left(\frac{1}{\pi} \int_0^\infty \frac{\log \psi(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta \right).$$

For a fixed ξ , the integrand on the right-hand side is a meromorphic function of ζ in the region $\operatorname{Re}(\xi \zeta) > 0$ with a simple pole at $-i$. Furthermore,

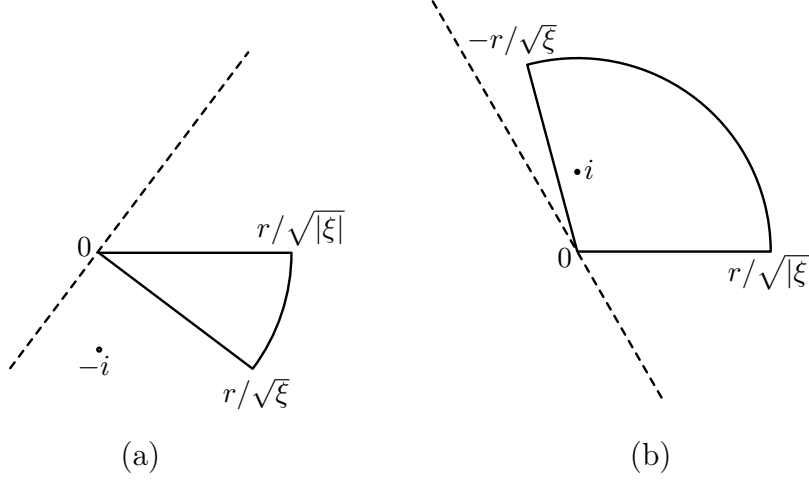


FIGURE 1. Two contours used in the proof of Lemma 3.8. Dashed line depicts the boundary of the region in which the integrand is meromorphic. In (a), $\operatorname{Re} \xi > 0$ and $\operatorname{Im} \xi > 0$, and the integrand $\log \psi(\xi^2 \zeta^2)/(1 + \zeta^2)$ is meromorphic in the region $\operatorname{Re}(\xi \zeta) > 0$ with pole at $-i$, outside the contour of integration. In (b), $\operatorname{Re} \xi < 0$ and $\operatorname{Im} \xi > 0$, and the integrand $\log \psi(\xi^2 \zeta^2)/(1 + \zeta^2)$ is meromorphic in $\operatorname{Re}(-\xi \zeta) > 0$ with a simple pole at i .

by Proposition 2.21(e), for any $\varepsilon > 0$ the integrand decays at least as fast as $|\zeta|^{-2} \log |\zeta|^2$ when $|\zeta| \rightarrow \infty$ in the region $-\pi/2 + \varepsilon < \operatorname{Arg}(\xi \zeta) < \pi/2 - \varepsilon$. Hence, by a standard contour integration (using contours shown in Figure 1 (a) with $r \rightarrow \infty$; the pole at $-i$ is outside the contour) and then parametrization $\zeta = s/\sqrt{\xi}$ of $[0, \xi^{-1/2}\infty)$, we obtain that

$$\begin{aligned} \psi^\dagger(\xi) &= \exp \left(\frac{1}{\pi} \int_0^{\xi^{-1/2}\infty} \frac{\log \psi(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta \right) \\ &= \exp \left(\frac{1}{\pi} \int_0^\infty \frac{\sqrt{\xi} \log \psi(\xi s^2)}{\xi + s^2} ds \right). \end{aligned}$$

The formula on the right-hand side clearly defines a holomorphic function in $\mathbf{C} \setminus (-\infty, 0]$ (integrability follows by Proposition 2.21(e)), which is the holomorphic extension of $\psi^\dagger(\xi)$. Suppose that $\operatorname{Re} \xi < 0$ and $\operatorname{Im} \xi > 0$. Using the parametrization $\zeta = -s/\sqrt{\xi}$ of $[0, -\xi^{-1/2}\infty)$, we obtain that

$$\psi^\dagger(\xi) = \exp \left(-\frac{1}{\pi} \int_0^{-\xi^{-1/2}\infty} \frac{\log \psi(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta \right).$$

For a fixed ξ , the integrand on the right-hand side is a meromorphic function of ζ in the region $\operatorname{Re}(-\xi \zeta) > 0$ with a simple pole at i . Furthermore, again by Proposition 2.21(e), for any $\varepsilon > 0$, the integrand decays at least as fast as $\zeta^{-2} \log |\zeta|^2$ when $|\zeta| \rightarrow \infty$ in the region $-\pi/2 + \varepsilon < \operatorname{Arg}(-\xi \zeta) < \pi/2 - \varepsilon$. Hence, again using standard contour integration (along contours shown in Figure 1 (b) with $r \rightarrow \infty$; this time the pole at i is inside the contour) and

the residue theorem, we obtain that

$$\begin{aligned}\psi^\dagger(\xi) &= \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\log \psi(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta + 2i \operatorname{Res}_i \left(\frac{\log \psi(\xi^2 \zeta^2)}{1 + \zeta^2} \right)\right) \\ &= \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\log \psi(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta + \log \psi(-\xi^2)\right) = \frac{\psi(-\xi^2)}{f(\xi^2)}.\end{aligned}$$

Recall that $\operatorname{Re} \xi < 0$ and $\operatorname{Im} \xi > 0$. Hence $\operatorname{Re}(-\xi) > 0$, so that $f(\xi^2) = f((- \xi)^2) = \psi^\dagger(-\xi)$. Formula (3.4) follows when $\operatorname{Re} \xi < 0$ and $\operatorname{Im} \xi > 0$.

Since ψ and f are complete Bernstein functions, by Corollary 2.16(c), $h_1(\xi) = \operatorname{Arg} \psi(-\xi^2)$ and $h_2(\xi) = \operatorname{Arg} f(\xi^2)$ are bounded harmonic functions in the region $\operatorname{Re} \xi < 0$, $\operatorname{Im} \xi > 0$, taking values in $[0, \pi]$ and $[-\pi, 0]$, respectively. Hence, h_1 and h_2 are Poisson integrals of their (bounded, measurable) boundary values g_1, g_2 (defined on $(-\infty, 0] \cup [0, i\infty)$). Furthermore, we have $g_1(is) = \operatorname{Arg} \psi(s^2) = 0$ and $g_2(-s) = \operatorname{Arg} f(s^2) = 0$ for $s > 0$, and $g_1(-s) \in [0, \pi]$, $g_2(is) \in [-\pi, 0]$ for almost all $s > 0$.

Observe that when $\operatorname{Re} \xi < 0$, $\operatorname{Im} \xi > 0$, we have

$$\operatorname{Arg} \psi^\dagger(\xi) = \operatorname{Arg} \psi(-\xi^2) - \operatorname{Arg} f(\xi^2) = h_1(\xi) - h_2(\xi).$$

Hence, $h(\xi) = \operatorname{Arg} \psi^\dagger(\xi)$ is a bounded harmonic function in the region $\operatorname{Re} \xi < 0$, $\operatorname{Im} \xi > 0$, with boundary value g given by $g = g_1$ on $(-\infty, 0]$, and $g = -g_2$ on $[0, i\infty)$. It follows that $g(\xi) \in [0, \pi]$ for almost every boundary point ξ , and so $h(\xi) \in [0, \pi]$ for all ξ such that $\operatorname{Re} \xi < 0$, $\operatorname{Im} \xi > 0$. It follows that $\operatorname{Im} \psi^\dagger(\xi) \geq 0$ in this region. We have already seen that $\operatorname{Im} \psi^\dagger(\xi) = \operatorname{Im} f(\xi^2) \geq 0$ when $\operatorname{Re} \xi > 0$, $\operatorname{Im} \xi > 0$. Finally, $\psi^\dagger(\bar{\xi}) = \overline{\psi^\dagger(\xi)}$, and so $\operatorname{Im} \psi^\dagger(\xi) \leq 0$ when $\operatorname{Im} \xi < 0$. By Corollary 2.16(c), ψ^\dagger is a CBF, and (3.4) extends to all $\xi \in \mathbf{C} \setminus \mathbf{R}$. \square

We note a third formula for (the holomorphic extension of) $\psi^\dagger(\xi)$, other than (3.1) and (3.2): when ψ is a complete Bernstein function, then

$$(3.5) \quad \psi^\dagger(\xi) = \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\sqrt{\xi} \log \psi(\xi s^2)}{\xi + s^2} ds\right), \quad \xi \in \mathbf{C} \setminus (-\infty, 0];$$

see the above proof of Lemma 3.8.

In Section 4, the transformation $\psi \mapsto \psi^\dagger$ is applied to the function

$$(3.6) \quad \psi_\lambda(\xi) = \frac{1 - \xi/\lambda^2}{1 - \psi(\xi)/\psi(\lambda^2)}, \quad \lambda > 0, \xi \in \mathbf{C} \setminus (-\infty, 0).$$

This definition is continuously extended at $\xi = \lambda^2$ by letting $\psi_\lambda(\lambda^2) = \psi(\lambda^2)/(\lambda^2 \psi'(\lambda^2))$. We denote $\psi_\lambda^\dagger(\xi) = (\psi_\lambda)^\dagger(\xi)$.

By Proposition 2.19, ψ_λ and $\log \psi_\lambda$ are complete Bernstein functions, and $\psi_\lambda(0) = 1$. It is easy to see that $\psi'_\lambda(\lambda) = (\psi(\lambda^2)|\psi''(\lambda^2)|)/(2\lambda^2(\psi'(\lambda^2))^2)$ (note that $\psi''(\lambda^2) < 0$). Since ψ is concave, its graph lies below the tangent line at $\xi = \lambda^2$, that is,

$$(3.7) \quad \psi_\lambda(\xi) \leq \frac{\psi(\lambda^2)}{\lambda^2 \psi'(\lambda^2)} + \frac{\psi(\lambda^2)|\psi''(\lambda^2)|}{2\lambda^2(\psi'(\lambda^2))^2} (\xi - \lambda^2), \quad \lambda, \xi > 0.$$

Formula (3.7) (plus $\xi - \lambda^2 \leq \xi$) combined with Proposition 3.5 yields that

$$(3.8) \quad \psi_\lambda^\dagger(\xi) \leq \left| \sqrt{\frac{\psi(\lambda^2)}{\lambda^2 \psi'(\lambda^2)}} + \sqrt{\frac{\psi(\lambda^2) |\psi''(\lambda^2)|}{2\lambda^2 (\psi'(\lambda^2))^2}} \xi \right|, \quad \lambda > 0, \operatorname{Re} \xi > 0.$$

We remark that more detailed estimates of ψ_λ and ψ_λ^\dagger can be found in [56].

4. DERIVATION OF THE FORMULA FOR EIGENFUNCTIONS

Below we prove Theorem 1.1. Our strategy is as follows. First we state a distributional version of the spectral problem for $\mathcal{A}_{(0,\infty)}$, and in Lemma 4.2 we prove that a distributional solution is automatically a strong solution. The (simpler) case of compound Poisson processes requires a different approach and is studied separately in Proposition 4.4. Next, we rephrase the distributional problem as a Wiener-Hopf equation, and we solve it using the Wiener-Hopf method. Finally, we combine the two parts and prove Theorem 1.1. In addition, we establish some basic properties of the eigenfunctions.

4.1. Distributional and strong eigenfunctions. In this part, X_t can be an arbitrary one-dimensional *symmetric* Lévy process with Lévy-Khintchine exponent Ψ . We define ψ by the formula $\Psi(\xi) = \psi(\xi^2)$. Note that $\psi(\xi)$ need not be a Bernstein function.

Recall that \mathcal{A} is the generator of X_t , and $\mathcal{A}_{(0,\infty)}$ is the generator of the process X_t killed upon leaving $(0, \infty)$, defined on an appropriate domain. Here we only consider the L^∞ , C_b and C_0 generators, formal definitions are given in Preliminaries. Recall also that $A \in \mathcal{S}'$ is the distributional generator of X_t , see Definition 2.4. The Fourier transform of the distribution A is $-\psi(\xi^2)$, and since A is symmetric, it is equal to its reflection \check{A} .

Let $F \in L^\infty((0, \infty))$. As usual, F is extended to entire real line by setting $F(x) = 0$ for $x \leq 0$. Given some regularity of X_t , if $F \in C_0((0, \infty))$ then, by Lemma 2.10, the condition $\mathcal{A}_{(0,\infty)} F = -\psi(\lambda^2) F$ is equivalent to the apparently weaker condition $A * F(x) = -\psi(\lambda^2) F(x)$ for $x \in (0, \infty)$. This motivates the following definition.

Definition 4.1. Let $\lambda > 0$. A tempered distribution F is said to be a *distributional eigenfunction* of $\mathcal{A}_{(0,\infty)}$, corresponding to the eigenvalue $-\psi(\lambda^2)$, if F is supported in $[0, \infty)$, \mathcal{S}' -convolvable with A , and $A * F + \psi(\lambda^2) F$ is supported in $(-\infty, 0]$. Informally,

$$(4.1) \quad \begin{aligned} F &= 0 && \text{on } (-\infty, 0), \\ A * F &= \psi(\lambda^2) F && \text{on } (0, \infty). \end{aligned}$$

Lemma 4.2. Suppose that X_t is a symmetric Lévy process, which has the strong Feller property, and that 0 is a regular boundary point of $(0, \infty)$. Let $\lambda > 0$, and suppose that F is a distributional eigenfunction of $\mathcal{A}_{(0,\infty)}$, corresponding to the eigenvalue $-\psi(\lambda^2)$. If F is bounded and continuous on \mathbf{R} , and if

$$\lim_{x \rightarrow \infty} (F(x) - C \sin(\lambda x + \vartheta)) = 0$$

for some $C, \vartheta \in \mathbf{R}$, then $F \in \mathcal{D}(\mathcal{A}_{(0,\infty)}; L^\infty)$ and $\mathcal{A}_{(0,\infty)}F = -\psi(\lambda^2)F$.

Proof. Since F does not vanish at infinity, we cannot apply Lemma 2.10 directly to F to prove that $F \in \mathcal{D}(\mathcal{A}_{(0,\infty)}; L^\infty)$. Thus, we decompose F into the sum of a $C_0((0, \infty))$ function and a smooth bounded function.

Denote $F^*(x) = C \sin(\lambda x + \vartheta)$. By Remark 2.1, $F^* \in \mathcal{D}(\mathcal{A}; L^\infty)$ and $\mathcal{A}F^* = -\psi(\lambda^2)F^*$. By Proposition 2.5, $A * F^* = -\psi(\lambda^2)F^*$. We choose an infinitely smooth function h such that $h(x) = 0$ for $x \leq 1$ and $h(x) = 1$ for x large enough. Furthermore, we require that

$$(4.2) \quad \int_1^\infty h(z)F^*(z)\nu(z)dz = 0.$$

(Note that such a choice is always possible, as we do not assume that $0 \leq h(x) \leq 1$.) We write $F = f_1 + f_2$, where $f_1 = hF^*$ and $f_2 = F - hF^*$. The function f_1 is infinitely smooth, with all derivatives bounded, so by Lemma 2.2, $f_1 \in \mathcal{D}(\mathcal{A}; C_b)$. We have $f_1(x) = 0$ for $x \leq 1$, and by (2.8) and (4.2), $\mathcal{A}f_1(0) = 0$. Hence, by Lemma 2.11 and Proposition 2.5, we have $f_1 \in \mathcal{D}(\mathcal{A}_{(0,\infty)}; L^\infty)$ and $\mathcal{A}_{(0,\infty)}f_1(x) = \mathcal{A}f_1(x) = A * f_1(x)$ for $x \in (0, \infty)$. We claim that also $f_2 \in \mathcal{D}(\mathcal{A}_{(0,\infty)}; L^\infty)$ and $\mathcal{A}_{(0,\infty)}f_2(x) = A * f_2(x)$ for $x \in (0, \infty)$. Once this is proved, we have $F \in \mathcal{D}(\mathcal{A}_{(0,\infty)}; L^\infty)$ and $\mathcal{A}_{(0,\infty)}F(x) = A * F(x) = -\psi(\lambda^2)F(x)$ for $x > 0$, as desired. Hence, it remains to prove the claim.

We will apply Lemma 2.10 to $f_2 = F - hF^*$. Since F and h vanish on $(-\infty, 0]$, we have $f_2(x) = 0$ for $x \leq 0$. Also, $f_2 = (F - F^*) + (1 - h)F^*$, and both $F - F^*$, $(1 - h)$ vanish at infinity, so that $\lim_{x \rightarrow \infty} f_2(x) = 0$. In other words, $f_2 \in C_0((0, \infty))$. Furthermore,

$$A * f_2 = A * F - A * (hF^*) = A * F - A * f_1.$$

By assumption, $A * F$ restricted to $(0, \infty)$ is equal to $-\psi(\lambda^2)F(x)$. We have already seen that $A * f_1 = \mathcal{A}f_1$ is a continuous function on \mathbf{R} which vanishes at 0. Hence, $A * f_2$ restricted to $(0, \infty)$ is equal to a continuous function, vanishing continuously at 0. Furthermore, for $x > 0$,

$$\begin{aligned} A * f_2(x) &= A * F(x) - A * F^*(x) + A * ((1 - h)F^*)(x) \\ &= -\psi(\lambda^2)(F(x) - F^*(x)) + A * ((1 - h)F^*)(x). \end{aligned}$$

By assumption, the first term on the right-hand side converges to 0 as $x \rightarrow \infty$. Since $1 - h(x) = 0$ for x large enough, also $A * ((1 - h)F^*)(x)$ vanishes as $x \rightarrow \infty$ (by Lemma 2.2 and Proposition 2.5). We conclude that $A * f_2$ restricted to $(0, \infty)$ is equal to a $C_0((0, \infty))$ function. By Lemma 2.10, $f_2 \in \mathcal{D}(\mathcal{A}_{(0,\infty)}; C_0)$ (and therefore also $f_2 \in \mathcal{D}(\mathcal{A}_{(0,\infty)}; L^\infty)$), and $\mathcal{A}_{(0,\infty)}f_2(x) = A * f_2(x)$ for $x > 0$. \square

Remark 4.3. Continuity of F at 0 is essential. Indeed, when X_t is the Brownian motion (i.e. $\psi(\xi) = \xi$), then $F(x) = \cos(\lambda x)\mathbf{1}_{(0,\infty)}(x)$ is the distributional eigenfunction of $\mathcal{A} = d^2/dx^2$ in $(0, \infty)$, but F is not in the $L^\infty(\mathbf{R}^d)$ domain of $\mathcal{A}_{(0,\infty)}$. In particular $P_t^{(0,\infty)}F$ is not equal to $e^{-\lambda^2 t}F$, as can be verified by a direct calculation. \square

When X_t is a compound Poisson process, the proof is much easier.

Proposition 4.4. *Suppose that X_t is a symmetric compound Poisson process. Let $\lambda > 0$, and suppose that $F \in L^\infty((0, \infty))$ is a distributional eigenfunction of $\mathcal{A}_{(0, \infty)}$, corresponding to the eigenvalue $-\psi(\lambda^2)$. Then $\mathcal{A}_{(0, \infty)}F = -\psi(\lambda^2)F$.*

Proof. The result is a straightforward application of Lemma 2.7 and the fact that A is the signed measure $\nu(dz) - \nu(\mathbf{R})\delta_0(dz)$. \square

4.2. Wiener-Hopf factorization. A method of solving problems of the form (4.1) was found in [72] in the case when A is an integrable function satisfying some growth conditions. For this reason, problems of this kind are called *Wiener-Hopf equations*, and the algorithm for finding their solutions is the *Wiener-Hopf method*; see [31, 46] for a detailed exposition. We begin with a brief description of the method. A version of Paley-Wiener theorem states that a function is supported in $[0, \infty)$ or $(-\infty, 0]$ if and only if its Fourier transform extends holomorphically to the upper or lower complex half-plane, respectively. Therefore, (4.1) can be re-written as

$\mathcal{F}F(\xi)$ extends to the upper complex half-plane,

$(C - \psi(\xi^2))\mathcal{F}F(\xi)$ extends to the lower complex half-plane,

where $C = \psi(\lambda^2)$. The main ingredient of the *Wiener-Hopf method* is factorization of the symbol $C - \psi(\xi^2)$ into the product of two functions, one extending holomorphically to the upper half-plane, the other one to the lower half-plane (the *Wiener-Hopf factors*). Then $\mathcal{F}F$ is taken to be the multiplicative inverse of the first factor.

The fundamental condition for the Wiener-Hopf method is that the symbol does not vanish. In our case, we have $C - \psi(\xi^2) = 0$ for $\xi = \pm\lambda$. For this reason, we apply the Wiener-Hopf method to the *regularized* symbol

$$\frac{1 - \psi(\xi^2)/\psi(\lambda^2)}{1 - \xi^2/\lambda^2} = \frac{1}{\psi_\lambda(\xi^2)};$$

here we use the notation of (3.6).

From now on, Assumption 2.15 is in force. It is required in order to establish some properties of $1/\psi_\lambda(\xi^2)$ and its Wiener-Hopf factors. In particular, since ψ_λ is a complete Bernstein function, the factorization is given by Lemma 3.8:

$$\frac{1}{\psi_\lambda(\xi^2)} = \frac{1}{\psi_\lambda^\dagger(-i\xi)} \frac{1}{\psi_\lambda^\dagger(i\xi)}, \quad \xi \in \mathbf{R}.$$

The Wiener-Hopf factors are $\psi_\lambda^\dagger(-i\xi)$ and $\psi_\lambda^\dagger(i\xi)$, and the solution is expected to satisfy $\mathcal{F}F(\xi) = \psi_\lambda^\dagger(-i\xi)/(1 - \xi^2/\lambda^2)$, up to a multiplicative constant. Due to the singularity in the denominator, we prefer to work with the Laplace transform of F . Below we provide details for the above idea.

Fix $\lambda > 0$, and define a tempered distribution

$$A_\lambda = \psi(\lambda^2)\delta_0 + A,$$

where A is the distributional generator of X_t and δ_0 is the Dirac delta measure. Hence, $\mathcal{F}A(\xi) = \psi(\lambda^2) - \psi(\xi^2)$. We decompose A_λ into the convolution of the distribution

$$(4.3) \quad S = \frac{\psi(\lambda^2)}{\lambda^2} (\lambda^2 \delta_0 + \delta_0'')$$

(the *singular part*) and two infinitely divisible measures R_+ and R_- (*regular parts*) supported in $[0, \infty)$ and $(-\infty, 0]$ respectively.

Clearly,

$$(4.4) \quad \mathcal{F}S(\xi) = \frac{\psi(\lambda^2)}{\lambda^2} (\lambda^2 - \xi^2) = \psi(\lambda^2) \left(1 - \frac{\xi^2}{\lambda^2}\right).$$

We define the tempered distribution R by means of Fourier transform,

$$(4.5) \quad \mathcal{F}R(\xi) = \frac{1 - \psi(\xi^2)/\psi(\lambda^2)}{1 - \xi^2/\lambda^2} = \frac{1}{\psi_\lambda(\xi^2)}, \quad \xi \in \mathbf{R},$$

with ψ_λ defined in (3.6). Since S is compactly supported, S and R are \mathcal{S}' -convolvable, and since $\mathcal{F}S(\xi)\mathcal{F}R(\xi) = \psi(\lambda^2) - \psi(\xi^2) = \mathcal{F}A_\lambda$, by the exchange formula we have

$$A_\lambda = S * R.$$

Lemma 4.5. *There is a probability measure R_+ supported in $[0, \infty)$, such that (see (3.2))*

$$(4.6) \quad \mathcal{L}R_+(\xi) = \exp \left(-\frac{1}{\pi} \int_0^\infty \frac{\xi \log \psi_\lambda(\zeta^2)}{\xi^2 + \zeta^2} d\zeta \right) = \frac{1}{\psi_\lambda^\dagger(\xi)}, \quad \operatorname{Re} \xi \geq 0.$$

Furthermore, R_+ has a completely monotone density on $(0, \infty)$ (but it may have an atom at 0).

If $R_-(E) = R_+(-E)$ is the reflection of R_+ , then $R = R_+ * R_-$ and $\mathcal{F}R(\xi) = \mathcal{F}R_+(\xi)\mathcal{F}R_-(\xi)$ ($\xi \in \mathbf{R}$).

Proof. By Lemma 3.8, ψ_λ^\dagger is a complete Bernstein function. By Proposition 2.17(a), $1/\psi_\lambda^\dagger$ is a Stieltjes function, and therefore it is the Laplace transform of a measure R_+ , which has completely monotone density function on $(0, \infty)$. By Proposition 3.4(d), $1/\psi_\lambda^\dagger(0) = 1/\psi_\lambda(0) = 1$. Hence, R_+ is a probability measure. Finally, $\mathcal{F}R_+(\xi) = 1/\psi_\lambda^\dagger(-i\xi)$ (where the holomorphic extension of ψ_λ^\dagger is denoted by the same symbol).

Let R_- be the reflection of R_+ . By (3.8),

$$\mathcal{F}R(\xi) = \frac{1}{\psi_\lambda(\xi^2)} = \frac{1}{\psi_\lambda^\dagger(-i\xi)} \frac{1}{\psi_\lambda^\dagger(i\xi)} = \mathcal{F}R_+(\xi)\mathcal{F}R_-(\xi)$$

for $\xi \in \mathbf{R}$. The lemma follows by the exchange formula. \square

Lemma 4.6. *Suppose that $F \in \mathcal{S}'$ is a distribution supported in $[0, \infty)$, for which $F * \varphi$ is bounded for any $\varphi \in \mathcal{S}$. If $F * (S * R_+)$ is supported in $\{0\}$, then F is a distributional eigenfunction of $\mathcal{A}_{(0, \infty)}$.*

Proof. Note that both F and $S * R_+$ are supported in $[0, \infty)$, and so they are \mathcal{S}' -convolvable. Furthermore, for any $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{S}$, $F * \varphi_1$ is bounded, and $(S * R_+) * \varphi_2, R_- * \varphi_3$ are integrable. Therefore, the \mathcal{S}' -convolution of $F, S * R_+$ and R_- is associative. In other words, if we let $\tilde{F} = F * (S * R_+)$, then $\tilde{F} * R_- = A_\lambda * F$. Clearly, if the support of \tilde{F} is $\{0\}$, then $A_\lambda * F = \tilde{F} * R_-$ is supported in $(-\infty, 0]$. \square

4.3. Formula for eigenfunctions. Lemma 4.6 enables us to describe a family of solutions to (4.1). If $F * (S * R_+)$ is supported in $\{0\}$, then $\mathcal{F}F \cdot \mathcal{F}(S * R_+) = P$ for some polynomial P . Therefore, the Fourier transform of a distributional eigenfunction F is expected to have the form $P/\mathcal{F}(S * R_+)$; however, extra care is needed because of the zeros of the denominator $\mathcal{F}(S * R_+)$ at $\pm\lambda$.

Let (see (4.6))

$$(4.7) \quad f(\xi) = \frac{c_\lambda}{(\lambda^2 + \xi^2)\mathcal{L}R_+(\xi)} = \frac{c_\lambda \psi_\lambda^\dagger(\xi)}{\lambda^2 + \xi^2} \quad \xi > 0,$$

where

$$(4.8) \quad c_\lambda = \sqrt{\frac{\lambda^2}{\psi_\lambda(\lambda^2)}} = \sqrt{\frac{\lambda^4 \psi'(\lambda^2)}{\psi(\lambda^2)}}$$

is a normalization constant. Since ψ_λ^\dagger is a complete Bernstein function, f extends to a meromorphic function in $\mathbf{C} \setminus (-\infty, 0]$, with poles at $\pm i\lambda$. We denote this extension again by f . By (3.2) and Proposition 3.4(b),

$$(4.9) \quad \begin{aligned} \psi_\lambda^\dagger(\xi) &= \sqrt{\psi_\lambda(\lambda^2)} \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + \zeta^2} \log \frac{\psi_\lambda(\zeta^2)}{\psi_\lambda(\lambda^2)} d\zeta\right) \\ &= \sqrt{\frac{\psi(\lambda^2)}{\lambda^2 \psi'(\lambda^2)}} \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + \zeta^2} \log \frac{\psi'(\lambda^2)(\lambda^2 - \zeta^2)}{\psi(\lambda^2) - \psi(\zeta^2)} d\zeta\right); \end{aligned}$$

this holds true for complex ξ with $\operatorname{Re} \xi > 0$. It follows that $f(\xi)$ is equal to the right-hand side of (1.1). In order to prove Theorem 1.1, we need to show that f is the Laplace transform of a function F of the form (1.2), and that F is a (distributional, and hence L^∞) eigenfunction of $\mathcal{A}_{(0,\infty)}$.

We define

$$(4.10) \quad f^*(\xi) = \frac{c_\lambda}{2i\lambda} \left(\frac{\psi_\lambda^\dagger(i\lambda)}{\xi - i\lambda} - \frac{\psi_\lambda^\dagger(-i\lambda)}{\xi + i\lambda} \right), \quad \xi \in \mathbf{C} \setminus \{-i\lambda, i\lambda\},$$

and

$$(4.11) \quad g(\xi) = f^*(\xi) - f(\xi) = \frac{1}{2i\lambda} \left(\frac{c_\lambda \psi_\lambda^\dagger(i\lambda)}{\xi - i\lambda} - \frac{c_\lambda \psi_\lambda^\dagger(-i\lambda)}{\xi + i\lambda} \right) - \frac{c_\lambda \psi_\lambda^\dagger(\xi)}{\lambda^2 + \xi^2}.$$

Note that $C_1(\xi - iC_2)^{-1}$ is the Laplace transform of $C_1 e^{-iC_2 x} \mathbf{1}_{(0,\infty)}(x)$. Hence, f^* is the Laplace transform of

$$(4.12) \quad F^*(x) = \frac{c_\lambda \psi_\lambda^\dagger(i\lambda) e^{i\lambda x} - c_\lambda \psi_\lambda^\dagger(-i\lambda) e^{-i\lambda x}}{2i\lambda} \mathbf{1}_{(0,\infty)}(x), \quad x \in \mathbf{R}.$$

By Lemma 2.20 (applied to a CBF $c_\lambda \psi_\lambda^\dagger$), g is a Stieltjes function, it is the Laplace transform of a completely monotone function $G(x)$, and G is the Laplace transform of a finite measure. Hence, f is the Laplace transform of $F(x) = F^*(x) - G(x)$.

Proposition 4.7. *We have $\psi_\lambda^\dagger(\pm i\lambda) = \sqrt{\psi_\lambda(\lambda^2)} e^{\pm i\vartheta_\lambda}$, where ϑ_λ is given by (1.3).*

Proof. By (4.9), for $\varepsilon \in (0, \lambda)$ we have

$$\psi_\lambda^\dagger(i\lambda + \varepsilon) = \sqrt{\psi_\lambda(\lambda^2)} \exp \left(\frac{1}{\pi} \int_0^\infty \frac{i\lambda + \varepsilon}{(i\lambda + \varepsilon)^2 + \zeta^2} \log \frac{\psi_\lambda(\zeta^2)}{\psi_\lambda(\lambda^2)} d\zeta \right).$$

We claim that the integrand on the right-hand side is dominated by an integrable function. Then, by dominated convergence,

$$\begin{aligned} \psi_\lambda^\dagger(i\lambda) &= \lim_{\varepsilon \rightarrow 0^+} \psi_\lambda^\dagger(i\lambda + \varepsilon) \\ &= \sqrt{\psi_\lambda(\lambda^2)} \exp \left(\frac{1}{\pi} \int_0^\infty \frac{i\lambda}{-\lambda^2 + \zeta^2} \log \frac{\psi_\lambda(\zeta^2)}{\psi_\lambda(\lambda^2)} d\zeta \right), \end{aligned}$$

as desired (see (1.3) and (4.9)). Hence, it remains to prove the claim.

Observe that

$$\begin{aligned} \left| \frac{i\lambda + \varepsilon}{(i\lambda + \varepsilon)^2 + \zeta^2} \right| &= \frac{|i\lambda + \varepsilon|}{|i(\lambda - \zeta) + \varepsilon| |i(\lambda + \zeta) + \varepsilon|} \\ &\leq \frac{\lambda + \varepsilon}{|\lambda - \zeta|(\lambda + \zeta)} \leq \frac{2\lambda}{|\lambda - \zeta|(\lambda + \zeta)}. \end{aligned}$$

Since ψ_λ is nonnegative and concave, we have $\psi_\lambda(\zeta^2)/\psi_\lambda(\lambda^2) \geq \zeta^2/\lambda^2$ for $\zeta \leq \lambda$, and $\psi_\lambda(\zeta^2)/\psi_\lambda(\lambda^2) \leq \zeta^2/\lambda^2$ for $\zeta \geq \lambda$. This and monotonicity of ψ_λ gives that

$$\left| \log \frac{\psi_\lambda(\zeta^2)}{\psi_\lambda(\lambda^2)} \right| \leq \left| \log \frac{\zeta^2}{\lambda^2} \right| = 2|\log \zeta - \log \lambda|.$$

It follows that

$$\left| \frac{i\lambda + \varepsilon}{(i\lambda + \varepsilon)^2 + \zeta^2} \log \frac{\psi_\lambda(\zeta^2)}{\psi_\lambda(\lambda^2)} \right| \leq \frac{4\lambda |\log \zeta - \log \lambda|}{|\zeta - \lambda|(\zeta + \lambda)},$$

which is integrable in $\zeta > 0$. \square

The above proposition and formulae (4.8) and (4.12) give that $F^*(x) = \sin(\lambda x + \vartheta_\lambda) \mathbf{1}_{(0, \infty)}(x)$. We summarize our development in the following result.

Corollary 4.8. *The function f is the Laplace transform of a function $F(x) = F^*(x) - G(x)$, where*

$$F^*(x) = \sin(\lambda x + \vartheta_\lambda) \mathbf{1}_{(0, \infty)}(x),$$

G is completely monotone on $(0, \infty)$ and it is the Laplace transform of a finite measure on $(0, \infty)$. \square

Lemma 4.9. *The function F constructed above is a distributional eigenfunction of $\mathcal{A}_{(0, \infty)}$.*

Proof. By (4.3), we have $\mathcal{L}S = (\psi(\lambda^2)/\lambda^2)(\lambda^2 + \xi^2)$. Since $F = F^* - G$ and both F^* and G are bounded, also F is a bounded function. Hence, F is S' -convolvable with $S * R_+$, and $\mathcal{L}F(\xi) \cdot (\mathcal{L}S(\xi)\mathcal{L}R_+(\xi)) = f(\xi) \cdot ((\psi(\lambda^2)/\lambda^2)(\lambda^2 + \xi^2)/\psi_\lambda^\dagger(\xi)) = c_\lambda \psi(\lambda^2)/\lambda^2$ when $\operatorname{Re} \xi > 0$. By the exchange formula, $F * (S * R_+) = (c_\lambda \psi(\lambda^2)/\lambda^2)\delta_0$, which is a distribution supported in $\{0\}$. An application of Lemma 4.6 completes the proof. \square

Remark 4.10. Clearly, also the distributional derivatives (of all orders) of F are distributional eigenfunctions of $\mathcal{A}_{(0,\infty)}$. However, they are not in the $L^\infty((0, \infty))$ domain of $\mathcal{A}_{(0,\infty)}$. \square

In order to apply Lemma 4.2, we need to show that F vanishes continuously at 0.

Lemma 4.11. *The function F has a nonnegative right limit at 0. If ψ is unbounded, then F vanishes continuously at 0.*

Proof. Since G is non-increasing (even completely monotone), the limit of $F(x)$ as $x \rightarrow 0^+$ exists. Since $\mathcal{L}F = f$, where f is given by (4.7), we have

$$\lim_{x \rightarrow 0^+} F(x) = \lim_{\xi \rightarrow \infty} \xi f(\xi) = \lim_{\xi \rightarrow \infty} \frac{c_\lambda \xi \psi_\lambda^\dagger(\xi)}{\lambda^2 + \xi^2} = \lim_{\xi \rightarrow \infty} \left(\frac{c_\lambda \xi^2}{\lambda^2 + \xi^2} \frac{\psi_\lambda^\dagger(\xi)}{\xi} \right).$$

The above expression is nonnegative, proving the first part. Furthermore, by Proposition 3.4(a,c), $\psi_\lambda^\dagger(\xi)/\xi = (\psi_\lambda(\xi)/\xi)^\dagger$. If ψ is unbounded on $[0, \infty)$, then $\psi_\lambda(\xi)/\xi$ tends to 0 as $\xi \rightarrow \infty$ (see (3.6)). Hence, by Proposition 3.4(d), $\psi_\lambda^\dagger(\xi)/\xi$ converges to 0 as $\xi \rightarrow \infty$, and therefore $\lim_{x \rightarrow 0^+} F(x) = 0$. \square

Proof of Theorem 1.1. In the remark following (4.9) it was already observed that $\mathcal{L}F(\xi) = f(\xi)$ coincides with the right-hand side of the formula (1.1) for $\mathcal{L}F_\lambda$, so that $F = F_\lambda$. We will show that F_λ satisfies also the remaining conditions of Theorem 1.1.

By Corollary 4.8, $F_\lambda = F^* - G$ has the form (1.2) with completely monotone $G_\lambda(x) = G(x)$, and G_λ is the Laplace transform of a finite measure γ_λ on $(0, \infty)$. By Lemma 4.9, F_λ is the distributional eigenfunction of $\mathcal{A}_{(0,\infty)}$.

If ψ is unbounded, then $F_\lambda = F$ vanishes continuously at 0 (Lemma 4.11), and $\lim_{x \rightarrow \infty} (F_\lambda(x) - \sin(\lambda x + \vartheta_\lambda)) = 0$ (because $G_\lambda = \mathcal{L}\gamma_\lambda$ tends to 0 at infinity). Furthermore, the subordinate Brownian motion X_t has the strong Feller property, and 0 is a regular boundary point of $(0, \infty)$ (see Preliminaries). Hence, Lemma 4.2 applies: $F_\lambda \in \mathcal{D}(\mathcal{A}_{(0,\infty)}; L^\infty)$ and $\mathcal{A}_{(0,\infty)}F_\lambda = -\psi(\lambda^2)F_\lambda$. When ψ is bounded, then $\mathcal{A}_{(0,\infty)}$ is a bounded operator on $L^\infty((0, \infty))$, and $\mathcal{A}_{(0,\infty)}F_\lambda = -\psi(\lambda^2)F_\lambda$ by Proposition 4.4. In both cases F_λ is an L^∞ eigenfunction of $\mathcal{A}_{(0,\infty)}$.

Note that (see [25])

$$P_t^{(0,\infty)}F(x) - F(x) = \int_0^t P_s^{(0,\infty)}\mathcal{A}_{(0,\infty)}F(x)ds = -\psi(\lambda^2) \int_0^t P_s^{(0,\infty)}f(x)ds$$

for $x \in (0, \infty)$. For a fixed x , this integral equation is easily solved, and we obtain $P_t^{(0,\infty)}F(x) = e^{-t\psi(\lambda^2)}F(x)$, that is, F_λ is the eigenfunction of $P_t^{(0,\infty)}$.

By Proposition 4.7, $\vartheta_\lambda = \text{Arg } \psi_\lambda^\dagger(i\lambda)$. Recall that $\psi_\lambda^\dagger(0) = 1$ (by Proposition 3.4(d)), and hence (by representation (2.23) for ψ_λ^\dagger) $\text{Re } \psi_\lambda^\dagger(i\lambda) \geq 1$. It follows that $0 \leq \vartheta_\lambda < \pi/2$, as desired. It remains to prove formula (1.4).

Suppose that (the holomorphic extension of) $\psi(\xi)$ extends to a continuous function $\psi^+(\xi)$ (not to be confused with $\psi^\dagger(\xi)$) in the region $\text{Im } \xi \geq 0$, and $\psi^+(-\xi) \neq \psi(\lambda^2)$ for all $\xi > 0$. Recall that $\mathcal{L}G_\lambda = g$ (with g given by (4.11)), and $G_\lambda = \mathcal{L}\gamma_\lambda$. Hence, by the remark following (2.24), $\gamma_\lambda = \pi^{-1}\tilde{m}_0$, where \tilde{m}_0 is the representation measure in (2.24) for the Stieltjes function g . To prove formula (1.4), it suffices to apply Proposition 2.18, as we now describe.

We have

$$g(\xi) = f^*(\xi) - f(\xi) = f^*(\xi) - \frac{c_\lambda \psi_\lambda^\dagger(\xi)}{\lambda^2 + \xi^2}.$$

By Lemma 3.8, when $\text{Im } \xi > 0$ and $\xi \neq i\lambda$,

$$g(\xi) = f^*(\xi) - \frac{c_\lambda \psi_\lambda(-\xi^2)}{\psi_\lambda^\dagger(-\xi)(\lambda^2 + \xi^2)} = f^*(\xi) - \frac{\psi(\lambda^2)}{\lambda^2 \psi_\lambda^\dagger(-\xi)(\psi(\lambda^2) - \psi(-\xi^2))}.$$

Hence, g extends to a continuous function g^+ in the region $\text{Im } \xi \geq 0$, and for $\xi > 0$,

$$g^+(-\xi) = f^*(-\xi) - \frac{c_\lambda \psi(\lambda^2)}{\lambda^2 \psi_\lambda^\dagger(\xi)(\psi(\lambda^2) - \psi^+(-\xi^2))}.$$

Note that $f^*(-\xi)$ and $\psi_\lambda^\dagger(\xi)$ are real for $\xi > 0$. By Proposition 2.18, $(-\text{Im } g^+(-\xi))d\xi$ is the measure \tilde{m}_0 in the representation (2.24) of the Stieltjes function g . Since $\gamma_\lambda = \pi^{-1}\tilde{m}_0$, $c_\lambda = \sqrt{\lambda^4 \psi'(\lambda^2)/\psi(\lambda^2)}$ and $\psi_\lambda(\lambda^2) = \psi(\lambda^2)/(\lambda^2 \psi'(\lambda^2))$, we obtain

$$\begin{aligned} \gamma_\lambda(d\xi) &= \frac{-\text{Im } g^+(-\xi)}{\pi} d\xi = \frac{1}{\pi} \frac{\sqrt{\psi'(\lambda^2)\psi(\lambda^2)}}{\psi_\lambda^\dagger(\xi)} \text{Im} \frac{1}{\psi(\lambda^2) - \psi^+(-\xi^2)} d\xi \\ &= \frac{1}{\pi} \frac{\sqrt{\psi_\lambda(\lambda^2)}}{\psi_\lambda^\dagger(\xi)} \text{Im} \frac{\lambda \psi'(\lambda^2)}{\psi(\lambda^2) - \psi^+(-\xi^2)} d\xi. \end{aligned}$$

An application of (4.9) completes the proof. \square

Remark 4.12. Using (4.9), formulae of Theorem 1.1 can be written in a more concise form. We record them for easier reference:

$$(4.13) \quad \vartheta_\lambda = \frac{1}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 - \zeta^2} \log \frac{\psi_\lambda(\zeta^2)}{\psi_\lambda(\lambda^2)} d\zeta = \text{Arg } \psi_\lambda^\dagger(i\lambda);$$

$$(4.14) \quad \mathcal{L}F_\lambda(\xi) = \frac{\lambda^2}{\lambda^2 + \xi^2} \frac{\psi_\lambda^\dagger(\xi)}{\sqrt{\psi_\lambda(\lambda^2)}};$$

$$(4.15) \quad \gamma_\lambda(d\xi) = \frac{1}{\pi} \frac{\sqrt{\psi_\lambda(\lambda^2)}}{\psi_\lambda^\dagger(\xi)} \text{Im} \frac{\lambda \psi'(\lambda^2)}{\psi(\lambda^2) - \psi^+(-\xi^2)} d\xi.$$

\square

Remark 4.13. The formula (1.4) for γ_λ remains true for general CBF ψ , as long as it is understood in the distributional sense. More precisely, it can be proved that γ_λ is the limit of

$$\frac{1}{\pi} \frac{\sqrt{\psi_\lambda(\lambda^2)}}{\psi_\lambda^\dagger(\xi)} \left(\operatorname{Im} \frac{\lambda \psi'(\lambda^2)}{\psi(\lambda^2) - \psi(-\xi^2 + i\varepsilon)} \right) \mathbf{1}_{(0,\infty)}(\xi) d\xi$$

as $\varepsilon \rightarrow 0^+$. We omit the details. \square

Remark 4.14. In order to extend Theorem 1.1 to more general symmetric Lévy processes, one needs two ingredients: formula (1.1) must define a function F_λ , and the convolution $F * (R_+ * S) * R_-$ needs to be associative (that is, some regularity of F and R_\pm is needed). This seems to be problematic, even for general subordinate Brownian motions. \square

4.4. Further properties of eigenfunctions. We begin with a technical result, useful for the computation and estimates of ϑ_λ .

Proposition 4.15. *We have*

$$(4.16) \quad \frac{1}{\pi} \int_0^1 \frac{-\log z}{1-z^2} dz = \frac{\pi}{8}.$$

Proof. By series expansion and integration by parts,

$$\begin{aligned} \int_0^1 \frac{-\log z}{1-z^2} dz &= \sum_{n=0}^{\infty} \int_0^1 z^{2n} (-\log z) dz = \sum_{n=0}^{\infty} \frac{1}{2n+1} \int_0^1 z^{2n} dz \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{3}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{3}{4} \frac{\pi^2}{6}, \end{aligned}$$

as desired. \square

The following formula for ϑ_λ is often more convenient than (1.3).

Proposition 4.16. *For $\lambda > 0$,*

$$(4.17) \quad \vartheta_\lambda = \frac{1}{\pi} \int_0^1 \frac{1}{1-z^2} \log \frac{\psi_\lambda(\lambda^2/z^2)}{\psi_\lambda(\lambda^2 z^2)} dz.$$

Proof. By substituting $\zeta = \lambda\xi$ in (1.3), we obtain

$$\vartheta_\lambda = -\frac{1}{\pi} \int_0^\infty \frac{1}{1-\xi^2} \log \frac{\lambda^2 \psi'(\lambda^2)(1-\xi^2)}{\psi(\lambda^2) - \psi(\lambda^2 \xi^2)} d\xi.$$

A substitution $\xi = z$ in the integral over $(0, 1)$ and $\xi = 1/z$ in the integral over $(1, \infty)$ yields

$$\begin{aligned} \vartheta_\lambda &= \frac{1}{\pi} \int_0^1 \frac{1}{1-z^2} \log \frac{\psi(\lambda^2) - \psi(\lambda^2 z^2)}{(\psi(\lambda^2/z^2) - \psi(\lambda^2)) z^2} dz \\ &= \frac{1}{\pi} \int_0^1 \frac{1}{1-z^2} \log \frac{\psi_\lambda(\lambda^2/z^2)}{\psi_\lambda(\lambda^2 z^2)} dz. \end{aligned} \quad \square$$

Proposition 4.17. *The function ϑ_λ is differentiable in $\lambda > 0$, and (4.17) can be differentiated in λ under the integral sign.*

Proof. Let $h(s, z) = (\log(\psi(s) - \psi(sz^2)) - \log(\psi(s/z^2) - \psi(s)) - 2 \log z$ for $z \in (0, 1)$ and $s > 0$. Then $h(\lambda^2, z) = \log(\psi_\lambda(\lambda^2 z^2)/\psi_\lambda(\lambda^2/z^2))$. Therefore, it suffices to prove that $(1 - z^2)^{-1} |(\partial/\partial s)h(s, z)|$ is integrable over rectangles $z \in (0, 1)$, $s \in (s_1, s_2)$, where $0 < s_1 < s_2$.

We have

$$\frac{\partial h(s, z)}{\partial s} = \frac{1}{s} \frac{s\psi'(s) - sz^2\psi'(sz^2)}{\psi(s) - \psi(sz^2)} - \frac{1}{s} \frac{sz^{-2}\psi'(s/z^2) - s\psi'(s)}{\psi(s/z^2) - \psi(s)}.$$

Using twice Cauchy's mean value theorem (for two functions $t\psi'(t)$ and $\psi(t)$), we obtain that for some $t_1 \in [sz^2, s]$ and $t_2 \in [s, s/z^2]$,

$$\frac{\partial h(s, z)}{\partial s} = \frac{1}{s} \frac{\psi'(t_1) + t_1\psi''(t_1)}{\psi'(t_1)} - \frac{1}{s} \frac{\psi'(t_2) + t_2\psi''(t_2)}{\psi'(t_2)}.$$

Since $t\psi''(t)/\psi'(t)$ is bounded (by Proposition 2.21(b)), the above expression is uniformly bounded in $s \in (s_1, s_2)$ and $z \in (0, 1)$. Furthermore, when $s \in (s_1, s_2)$ and $z \in (1/2, 1)$, by the usual mean value theorem,

$$\begin{aligned} \left| \frac{\partial h(s, z)}{\partial s} \right| &\leq \frac{t_2 - t_1}{s} \sup_{t \in [sz^2, s/z^2]} \left| \frac{d}{dt} \frac{\psi'(t) + t\psi''(t)}{\psi'(t)} \right| \\ &\leq \left(\frac{1}{z^2} - z^2 \right) \sup_{t \in [s_1/4, 4s_2]} \left| \frac{d}{dt} \frac{\psi'(t) + t\psi''(t)}{\psi'(t)} \right| \leq C(1 - z^2) \end{aligned}$$

for some C depending on s_1 and s_2 . The proof is complete. \square

Proposition 4.18. *We have*

$$(4.18) \quad \vartheta_\lambda \leq \sup_{\xi \in \mathbf{R}} \frac{\xi \psi'_\lambda(\xi)}{\psi_\lambda(\xi)} \cdot \frac{\pi}{2}.$$

Proof. Note that $(d/ds) \log \psi_\lambda(\lambda^2 s) = \lambda^2 \psi'_\lambda(\lambda^2 s)/\psi_\lambda(\lambda^2 s)$. Hence, by (4.17),

$$\vartheta_\lambda = \frac{1}{\pi} \int_0^1 \frac{1}{1 - z^2} \left(\int_{z^2}^{1/z^2} \frac{\lambda^2 \psi'_\lambda(\lambda^2 s)}{\psi_\lambda(\lambda^2 s)} ds \right) dz.$$

It follows that

$$\vartheta_\lambda \leq \left(\sup_{\xi \in \mathbf{R}} \frac{\xi \psi'_\lambda(\xi)}{\psi_\lambda(\xi)} \right) \frac{1}{\pi} \int_0^1 \frac{1}{1 - z^2} \left(\int_{z^2}^{1/z^2} \frac{1}{s} ds \right) dz.$$

Finally, by (4.16),

$$\frac{1}{\pi} \int_0^1 \frac{1}{1 - z^2} \left(\int_{z^2}^{1/z^2} \frac{1}{s} ds \right) dz = \frac{4}{\pi} \int_0^1 \frac{-\log z}{1 - z^2} dz = \frac{\pi}{2}. \quad \square$$

Corollary 4.19. *If $\vartheta_\lambda = 0$ for some $\lambda > 0$, then ψ is linear, X_t is a Brownian motion, and $\vartheta_\lambda = 0$ for all $\lambda > 0$.*

Proof. Since ψ_λ is increasing, for $z \in (0, 1)$ we have $\psi_\lambda(\lambda^2/z^2) \geq \psi_\lambda(\lambda^2 z^2)$, and it is easily checked that equality holds if and only if ψ is linear. Hence, by (4.17), $\vartheta_\lambda \geq 0$, and if ψ is not linear, then $\vartheta_\lambda > 0$. \square

Now we turn to the properties of F_λ .

Proposition 4.20. *We have*

$$(4.19) \quad \lim_{\xi \rightarrow 0^+} \mathcal{L}F_\lambda(\xi) = \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}}, \quad \lambda > 0.$$

Proof. The result follows from $\psi_\lambda^\dagger(0) = 1$ (see Proposition 3.4(d)), $\mathcal{L}F_\lambda(\xi) = c_\lambda \psi_\lambda^\dagger(\xi)/(\lambda^2 + \xi^2)$, and (4.8). \square

Lemma 4.21. *For all $\lambda, x > 0$, $0 \leq G_\lambda(x) \leq \sin \vartheta_\lambda$. Furthermore,*

$$(4.20) \quad \int_0^\infty G_\lambda(x) dx = \frac{1}{\lambda} \left(\cos \vartheta_\lambda - \sqrt{\frac{\lambda^2 \psi'(\lambda^2)}{\psi(\lambda^2)}} \right), \quad \lambda > 0.$$

Proof. Clearly, $0 \leq G_\lambda(x) \leq G_\lambda(0)$, and by Lemma 4.11, we have $0 \leq \lim_{x \rightarrow 0^+} F_\lambda(x) = \sin \vartheta_\lambda - G_\lambda(0)$. This proves the first statement. Furthermore, $\mathcal{L}G_\lambda(\xi) = g(\xi) = f^*(\xi) - f(\xi) = f^*(\xi) - \mathcal{L}F_\lambda(\xi)$. By (4.8), (4.10), (4.19) and Proposition 4.7,

$$\begin{aligned} \int_0^\infty G_\lambda(x) dx &= \lim_{\xi \rightarrow 0^+} \mathcal{L}G_\lambda(\xi) = f^*(0) - \lim_{\xi \rightarrow 0^+} \mathcal{L}F_\lambda(\xi) \\ &= \frac{c_\lambda \operatorname{Re} \psi_\lambda^\dagger(i\lambda)}{\lambda^2} - \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} = \frac{\cos \vartheta_\lambda}{\lambda} - \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}}. \end{aligned} \quad \square$$

Proposition 4.22. *We have*

$$(4.21) \quad |\mathcal{L}F_\lambda(\xi)| \leq \frac{|\lambda + \xi|}{|\lambda^2 + \xi^2|}, \quad \lambda > 0, \operatorname{Re} \xi > 0.$$

Proof. Since $\mathcal{L}F_\lambda(\xi) = c_\lambda \psi_\lambda^\dagger(\xi)/(\lambda^2 + \xi^2)$, by (3.8) and (4.8) we have

$$|\mathcal{L}F_\lambda(\xi)| \leq \frac{\lambda}{|\lambda^2 + \xi^2|} \left| 1 + \sqrt{\frac{|\psi''(\lambda^2)|}{2\psi'(\lambda^2)}} \xi \right|.$$

By Proposition 2.21(b), $\lambda^2 |\psi''(\lambda^2)| \leq 2\psi'(\lambda^2)$, and (4.21) follows. \square

Lemma 4.23. *Suppose that there are $C, \alpha > 0$ such that $\psi'(\xi) \geq C\xi^{\alpha-1}$ for ξ large enough. Then (cf. (4.21))*

$$(4.22) \quad |\mathcal{L}F_\lambda(\xi)| \leq \sqrt{\sup_{\zeta > \lambda} \frac{\lambda^{2-2\alpha} \psi'(\lambda^2)}{\zeta^{2-2\alpha} \psi'(\zeta^2)}} \frac{\lambda}{|\lambda^2 + \xi^2|} \left| 1 + \frac{\xi}{\lambda} \right|^{1-\alpha}$$

whenever $\lambda > 0$, and $\operatorname{Re} \xi > 0$.

Proof. Since ψ is concave and increasing, for $\lambda, \xi > 0$ we have

$$\psi_\lambda(\xi) = \frac{\psi(\lambda^2)}{\lambda^2} \frac{\lambda^2 - \xi}{\psi(\lambda^2) - \psi(\xi)} \leq \frac{\psi(\lambda^2)}{\lambda^2} \max \left(\frac{1}{\psi'(\lambda^2)}, \frac{1}{\psi'(\xi)} \right).$$

Let C_λ be the supremum in (4.22) (finite by the assumption), so that $\psi'(\lambda^2)/\psi'(\xi) \leq C_\lambda(\xi/\lambda^2)^{1-\alpha}$ for all $\xi > \lambda^2$. Then

$$\psi_\lambda(\xi) \leq \frac{C_\lambda \psi(\lambda^2)}{\lambda^2 \psi'(\lambda^2)} \left(1 + \frac{\xi}{\lambda^2} \right)^{1-\alpha}.$$

By Proposition 3.5, we have

$$(4.23) \quad \psi_\lambda^\dagger(\xi) \leq \sqrt{\frac{C_\lambda \psi(\lambda^2)}{\lambda^2 \psi'(\lambda^2)}} \left| 1 + \frac{\xi}{\lambda} \right|^{1-\alpha}, \quad \lambda > 0, \operatorname{Re} \xi > 0.$$

The lemma follows from $\mathcal{L}F_\lambda(\xi) = c_\lambda \psi_\lambda^\dagger(\xi)/(\xi^2 + \lambda^2)$ and (4.8). \square

Lemma 4.24. *Suppose that there are $C, \alpha > 0$ such that $\psi'(\xi) \geq C\xi^{\alpha-1}$ for sufficiently large ξ . Then for each $\lambda > 0$, the function F_λ is Hölder continuous with any exponent less than α . More precisely, there is an absolute constant c such that*

$$(4.24) \quad \frac{|F_\lambda(x_1) - F_\lambda(x_2)|}{|x_1 - x_2|^\varepsilon} \leq \frac{c\lambda^\varepsilon}{\alpha - \varepsilon} \sqrt{\sup_{\zeta > \lambda} \frac{\lambda^{2-2\alpha} \psi'(\lambda^2)}{\zeta^{2-2\alpha} \psi'(\zeta^2)}}$$

for all $\varepsilon \in (0, \alpha)$, $\lambda > 0$ and $x_1, x_2 \in \mathbf{R}$.

Proof. We decompose $\mathcal{L}F_\lambda = c_\lambda \psi_\lambda^\dagger(\xi)/(\lambda^2 + \xi^2)$ into the difference of \tilde{f}^* and \tilde{g} in a similar way to the decomposition $\mathcal{L}F_\lambda = f^* - g$ in the derivation of the formula for F_λ , but this time \tilde{f}^* and \tilde{g} have a better decay rate at infinity. When $\operatorname{Re} \xi > 0$, define

$$\begin{aligned} \tilde{f}^*(\xi) &= \frac{c_\lambda}{2i\lambda} \left(\frac{\lambda \psi_\lambda^\dagger(i\lambda)}{(\xi - i\lambda)(\xi - i\lambda + \lambda)} - \frac{\lambda \psi_\lambda^\dagger(-i\lambda)}{(\xi + i\lambda)(\xi + i\lambda + \lambda)} \right), \\ \tilde{g}(\xi) &= \frac{c_\lambda}{2i\lambda} \left(\frac{\lambda \psi_\lambda^\dagger(i\lambda) - (\xi - i\lambda + \lambda) \psi_\lambda^\dagger(\xi)}{(\xi - i\lambda)(\xi - i\lambda + \lambda)} \right. \\ &\quad \left. - \frac{\lambda \psi_\lambda^\dagger(-i\lambda) - (\xi + i\lambda + \lambda) \psi_\lambda^\dagger(\xi)}{(\xi + i\lambda)(\xi + i\lambda + \lambda)} \right). \end{aligned}$$

Note that $\lambda(\xi \mp i\lambda)^{-1}(\xi \mp i\lambda + \lambda)^{-1}$ is the Laplace transform of the function $(1 - e^{-\lambda x})e^{\pm i\lambda x} \mathbf{1}_{(0, \infty)}(x)$, and $c_\lambda \psi_\lambda^\dagger(\pm i\lambda) = \lambda e^{\pm i\vartheta_\lambda}$. Therefore, \tilde{f}^* is the Laplace transform of

$$\tilde{F}^*(x) = (1 - e^{-\lambda x}) \sin(\lambda x + \vartheta_\lambda) \mathbf{1}_{(0, \infty)}(x).$$

Note that \tilde{F}^* is Lipschitz continuous on \mathbf{R} . In fact, for $x > 0$ we have

$$\begin{aligned} |(\tilde{F}^*)'(x)| &= |\lambda e^{-\lambda x} \sin(\lambda x + \vartheta_\lambda) + \lambda(1 - e^{-\lambda x}) \cos(\lambda x + \vartheta_\lambda)| \\ &\leq \sqrt{\lambda^2 e^{-2\lambda x} + \lambda^2 (1 - e^{-\lambda x})^2} \leq \lambda, \end{aligned}$$

so that the Lipschitz constant of \tilde{F}^* is not greater than λ .

The function \tilde{g} extends continuously to $i\mathbf{R}$. Denote the summands in the parentheses in the definition of \tilde{g} by \tilde{g}_+ and \tilde{g}_- , so that $\tilde{g}(\xi) = (c_\lambda/(2i\lambda))(\tilde{g}_+(\xi) - \tilde{g}_-(\xi))$. We first consider $\tilde{g}_+(i\xi)$ for $\xi \in [\lambda/2, 2\lambda]$. We

have

$$\begin{aligned}
|\tilde{g}_+(i\xi)| &= \left| \frac{\lambda\psi_\lambda^\dagger(i\lambda) - (i\xi - i\lambda + \lambda)\psi_\lambda^\dagger(i\xi)}{(i\xi - i\lambda)(i\xi - i\lambda + \lambda)} \right| \\
&\leq \frac{\lambda|\psi_\lambda^\dagger(i\lambda) - \psi_\lambda^\dagger(i\xi)| + |\xi - \lambda||\psi_\lambda^\dagger(i\xi)|}{|\xi - \lambda|\sqrt{(\xi - \lambda)^2 + \lambda^2}} \\
&\leq \frac{1}{\sqrt{(\xi - \lambda)^2 + \lambda^2}} \left(\lambda \sup_{\zeta \in [\lambda, \xi]} |(\psi_\lambda^\dagger)'(i\zeta)| + |\psi_\lambda^\dagger(i\xi)| \right);
\end{aligned}$$

when $\lambda > \xi$, then we understand that $[\lambda, \xi]$ denotes the interval $[\xi, \lambda]$. By Proposition 2.21(c,d), $|(\psi_\lambda^\dagger)'(i\zeta)| \leq 2\psi_\lambda^\dagger(\zeta)/\zeta$ and $|\psi_\lambda^\dagger(i\zeta)| \leq 2\psi_\lambda^\dagger(\zeta)$ for $\zeta \in \mathbf{R}$; the latter inequality is used frequently in the remainder of the proof. Using also $\sqrt{(\xi - \lambda)^2 + \lambda^2} \geq \lambda$, we obtain

$$|\tilde{g}_+(i\xi)| \leq \frac{1}{\lambda} \left(\lambda \sup_{\zeta \in [\lambda/2, 2\lambda]} \frac{2\psi_\lambda^\dagger(\zeta)}{\zeta} + 2\psi_\lambda^\dagger(\xi) \right) \leq \frac{6\psi_\lambda^\dagger(2\lambda)}{\lambda}.$$

By (4.23), $c_\lambda\psi_\lambda^\dagger(2\lambda) \leq 3^{1-\alpha}\lambda\sqrt{C_\lambda} \leq 3\lambda\sqrt{C_\lambda}$, where C_λ is the supremum in (4.22). Hence,

$$|c_\lambda\tilde{g}_+(i\xi)| \leq 18\sqrt{C_\lambda}, \quad \lambda > 0, \xi \in [\lambda/2, 2\lambda].$$

When $\xi \in [-2\lambda, \lambda/2]$, we simply have $|\xi - \lambda| \geq \lambda/2$, $\sqrt{(\xi - \lambda)^2 + \lambda^2} \geq \lambda$, so that

$$|\tilde{g}_+(i\xi)| \leq \frac{\lambda|\psi_\lambda^\dagger(i\lambda)|}{|\xi - \lambda|\sqrt{(\xi - \lambda)^2 + \lambda^2}} + \frac{|\psi_\lambda^\dagger(i\xi)|}{|\xi - \lambda|} \leq \frac{4\psi_\lambda^\dagger(\lambda) + 4\psi_\lambda^\dagger(2\lambda)}{\lambda}.$$

Using (4.23), we obtain that $c_\lambda\psi_\lambda^\dagger(\lambda) \leq 2^{1-\alpha}\lambda\sqrt{C_\lambda} \leq 2\lambda\sqrt{C_\lambda}$. Together with $c_\lambda\psi_\lambda^\dagger(2\lambda) \leq 3\lambda\sqrt{C_\lambda}$, this gives

$$|c_\lambda\tilde{g}_+(i\xi)| \leq 20\sqrt{C_\lambda}, \quad \lambda > 0, \xi \in [-2\lambda, \lambda/2].$$

Similar estimates hold for $c_\lambda\tilde{g}_-$. It follows that

$$|\tilde{g}(i\xi)| \leq \frac{c_\lambda(|\tilde{g}_+(\xi)| + |\tilde{g}_-(\xi)|)}{2\lambda} \leq \frac{20\sqrt{C_\lambda}}{\lambda}, \quad \lambda > 0, \xi \in [-2\lambda, 2\lambda].$$

The estimate for $\xi \in \mathbf{R} \setminus [-2\lambda, 2\lambda]$ is obtained using $|\tilde{g}(i\xi)| \leq |\mathcal{L}F_\lambda(i\xi)| + |\tilde{f}^*(i\xi)|$ (here $\mathcal{L}F_\lambda(i\xi)$ denotes the continuous boundary limit). By the inequalities $|\xi - \lambda| \geq |\xi|/2$, $\sqrt{(\xi - \lambda)^2 + \lambda^2} \geq |\xi|/2$, we have

$$\begin{aligned}
|\tilde{f}^*(i\xi)| &\leq \frac{c_\lambda}{2} \left(\frac{|\psi_\lambda^\dagger(i\lambda)|}{|\xi - \lambda|\sqrt{(\xi - \lambda)^2 + \lambda^2}} + \frac{|\psi_\lambda^\dagger(-i\lambda)|}{|\xi + \lambda|\sqrt{(\xi + \lambda)^2 + \lambda^2}} \right) \\
&\leq \frac{8c_\lambda\psi_\lambda^\dagger(\lambda)}{\xi^2} \leq \frac{16\lambda\sqrt{C_\lambda}}{\xi^2}.
\end{aligned}$$

Hence, using also (4.22) and the inequality $\xi^2 - \lambda^2 \geq (|\xi| + \lambda)^2/3$, for $\xi \in \mathbf{R} \setminus [-2\lambda, 2\lambda]$ we obtain

$$\begin{aligned} |\tilde{g}(i\xi)| &\leq |\mathcal{L}F_\lambda(i\xi)| + |\tilde{f}^*(i\xi)| \leq \frac{\lambda\sqrt{C_\lambda}(1 + |\xi|/\lambda)^{1-\alpha}}{\xi^2 - \lambda^2} + \frac{16\lambda\sqrt{C_\lambda}}{\xi^2} \\ &\leq \frac{3\sqrt{C_\lambda}}{\lambda(1 + |\xi|/\lambda)^{1+\alpha}} + \frac{16\sqrt{C_\lambda}}{\lambda(|\xi|/\lambda)^2} \leq \frac{20\sqrt{C_\lambda}}{\lambda} \left(\frac{\lambda}{|\xi|} \right)^{1+\alpha}. \end{aligned}$$

This way we have estimated $|\tilde{g}(i\xi)|$ for all $\xi \in \mathbf{R}$. The function \tilde{g} is the Laplace transform of \tilde{G} , and \tilde{G} is the inverse Fourier transform of $\tilde{g}(-i\xi)$ ($\xi \in \mathbf{R}$). Fix $\varepsilon \in (0, \alpha)$. We have

$$\begin{aligned} |\tilde{G}(x_1) - \tilde{G}(x_2)| &= \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} (e^{-ix_1\xi} - e^{-ix_2\xi}) \tilde{g}(-i\xi) d\xi \right| \\ &\leq \frac{1}{2\pi} \sup_{\xi \in \mathbf{R}} \frac{|e^{-ix_1\xi} - e^{-ix_2\xi}|}{|\xi|^\varepsilon} \int_{-\infty}^{\infty} \xi^\varepsilon |\tilde{g}(-i\xi)| d\xi. \end{aligned}$$

Since $|e^{-ix_1\xi} - e^{-ix_2\xi}| \leq \min(2, |\xi||x_1 - x_2|)$, the supremum is at most $2^{1-\varepsilon}|x_1 - x_2|^\varepsilon \leq 2|x_1 - x_2|^\varepsilon$. Furthermore,

$$\begin{aligned} \int_{-\infty}^{\infty} \xi^\varepsilon |\tilde{g}(-i\xi)| d\xi &\leq \frac{20\sqrt{C_\lambda}}{\lambda} \left(2 \int_0^{2\lambda} \xi^\varepsilon d\xi + 2 \int_{2\lambda}^{\infty} \frac{\lambda^{1+\alpha}}{\xi^{1+\alpha-\varepsilon}} d\xi \right) \\ &= 20\sqrt{C_\lambda} \left(\frac{2^{2+\varepsilon}\lambda^\varepsilon}{1+\varepsilon} + \frac{2^{1-\alpha+\varepsilon}\lambda^\varepsilon}{\alpha-\varepsilon} \right) \\ &= 20\sqrt{C_\lambda} \left(8 + \frac{2}{\alpha-\varepsilon} \right) \lambda^\varepsilon \leq \frac{200\sqrt{C_\lambda}\lambda^\varepsilon}{\alpha-\varepsilon}. \end{aligned}$$

In particular, \tilde{G} is Hölder continuous of order ε , with Hölder constant at most $(200/\pi)\lambda^\varepsilon\sqrt{C_\lambda}/(\alpha-\varepsilon)$. The lemma is proved. \square

Corollary 4.25. *Suppose that for some $\varepsilon > 0$,*

$$(4.25) \quad \limsup_{\xi \rightarrow 0^+} \frac{\xi|\psi''(\xi)|}{\psi'(\xi)} < 1 - \varepsilon, \quad \limsup_{\xi \rightarrow \infty} \frac{\xi|\psi''(\xi)|}{\psi'(\xi)} < 1 - \varepsilon.$$

(see (1.11)). Then there is $C > 0$ (depending on ε) such that

$$|F_\lambda(x_1) - F_\lambda(x_2)| \leq C\lambda^\varepsilon|x_1 - x_2|^\varepsilon, \quad \lambda > 0, x_1, x_2 \in \mathbf{R},$$

and

$$|\mathcal{L}F_\lambda(\xi)| \leq C \frac{\lambda}{\lambda^2 + \xi^2} \left| 1 + \frac{\xi}{\lambda} \right|^{1-\varepsilon}, \quad \lambda > 0, \operatorname{Re} \xi > 0.$$

Note that in general, $\xi|\psi''(\xi)| \leq 2\psi'(\xi)$, see Proposition 2.21(b).

Proof. Choose $\alpha > \varepsilon$ small enough, so that (4.25) holds with ε replaced by α . Let $h(\zeta) = \zeta^{1-\alpha}\psi'(\zeta)$. For $\xi \geq \lambda > 0$ we have

$$\begin{aligned} \frac{\lambda^{2-2\alpha}\psi'(\lambda^2)}{\xi^{2-2\alpha}\psi'(\xi^2)} &= \frac{h(\lambda^2)}{h(\xi^2)} = \exp\left(-\int_{\lambda^2}^{\xi^2} \frac{h'(\zeta)}{h(\zeta)} d\zeta\right) \\ &= \exp\left(-\int_{\lambda^2}^{\xi^2} \left(\frac{1-\alpha}{\zeta} - \frac{|\psi''(\zeta)|}{\psi'(\zeta)}\right) d\zeta\right). \end{aligned}$$

The integrand on the right-hand side is positive for $\zeta \in (0, C_1) \cup (C_2, \infty)$ for some $C_1, C_2 > 0$. Hence the above expression is bounded uniformly in λ and ξ , $0 < \lambda \leq \xi$. The result follows by Lemmas 4.23 and 4.24. \square

Recall that a function f is said to be *regularly varying* of order $\alpha > 0$ at ∞ or at 0, if for all $k > 0$, $\lim_{x \rightarrow \infty} f(kx)/f(x) = k^\alpha$ or $\lim_{x \rightarrow 0^+} f(kx)/f(x) = k^\alpha$, respectively.

Corollary 4.26. *Suppose that ψ is regularly varying of order $\alpha_\infty > 0$ at ∞ and regularly varying of order $\alpha_0 > 0$ at 0. Then for all $\varepsilon < \min(\alpha_0, \alpha_\infty)$, the hypothesis of Corollary 4.25 holds true.*

Proof. Let $\varepsilon < \alpha < \min(\alpha_0, \alpha_\infty)$. Since ψ is a regularly varying Bernstein function, also ψ' and ψ'' are regularly varying at 0 and ∞ (see [6]). Hence, if we define $h(\zeta) = \zeta^{1-\alpha}\psi'(\zeta)$, then h is regularly varying of order $\alpha_0 - \alpha > 0$ at zero, and regularly varying of order $\alpha_\infty - \alpha > 0$ at ∞ , and a similar statement with orders decreased by 1 is true for h' . It follows that the function $\zeta h'(\zeta)/h(\zeta)$ (continuous on $(0, \infty)$) has positive limits at 0 and ∞ , and we can now repeat the proof of Corollary 4.25. \square

When ψ is regularly varying at infinity, then the behavior of F_λ near 0 for a fixed $\lambda > 0$ has a very simple description. We write $f(x) \sim g(x)$ as $x \rightarrow 0^+$ or $x \rightarrow \infty$ when $f(x)/g(x)$ converges to 1 as $x \rightarrow 0^+$ or $x \rightarrow \infty$, respectively.

Lemma 4.27. *If ψ is unbounded in $(0, \infty)$ and $\psi(\xi)$ is regularly varying at infinity of order $\alpha \in [0, 1]$, then*

$$(4.26) \quad \mathcal{L}F_\lambda(\xi) \sim \sqrt{\frac{\lambda^2\psi'(\lambda^2)}{\xi^2\psi(\xi^2)}} \quad \text{as } \xi \rightarrow \infty,$$

$$(4.27) \quad F_\lambda(x) \sim \frac{1}{\Gamma(1+\alpha)} \sqrt{\frac{\lambda^2\psi'(\lambda^2)}{\psi(x^{-2})}} \quad \text{as } x \rightarrow 0^+.$$

Proof. Consider an auxiliary function:

$$h(\xi, \zeta) = \frac{\lambda^2\psi(\xi^2)}{\psi(\lambda^2)\zeta^{2-2\alpha}\xi^2} \psi_\lambda(\xi^2\zeta^2), \quad \xi, \zeta > 0.$$

Since ψ_λ is a complete Bernstein function, both $\psi_\lambda(\xi)$ and $\xi/\psi_\lambda(\xi)$ are increasing (Proposition 2.17(a)). It follows that for $\xi > 0$, $\zeta \geq 1$, we have

$\psi_\lambda(\xi^2) \leq \psi_\lambda(\xi^2 \zeta^2) \leq \zeta^2 \psi_\lambda(\xi^2)$. Hence

$$(4.28) \quad \frac{\lambda^2}{\psi(\lambda^2)\zeta^{2-2\alpha}} \frac{\psi(\xi^2)\psi_\lambda(\xi^2)}{\xi^2} \leq h(\xi, \zeta) \leq \frac{\lambda^2}{\psi(\lambda^2)\zeta^{-2\alpha}} \frac{\psi(\xi^2)\psi_\lambda(\xi^2)}{\xi^2}.$$

Note that $\psi(\xi^2)\psi_\lambda(\xi^2)/\xi^2$ has a positive limit $\psi(\lambda^2)/\lambda^2$ as $\xi \rightarrow \infty$. Hence, it is bounded above and below by positive constants when $\xi \in (1, \infty)$. It follows that for some $C > 0$ (depending on λ),

$$|\log h(\xi, \zeta)| \leq C(1 + |\log \zeta|) \quad \xi > 1, \zeta \geq 1.$$

For $\zeta \in (0, 1]$, the inequalities in (4.28) are reversed, and we obtain the same bound for $|\log h(\xi, \zeta)|$. Since ψ is regularly varying of order α and unbounded,

$$\begin{aligned} \lim_{\xi \rightarrow \infty} h(\xi, \zeta) &= \lim_{\xi \rightarrow \infty} \frac{\psi(\xi^2)}{\zeta^{2-2\alpha}\xi^2} \frac{\lambda^2 - \xi^2\zeta^2}{\psi(\lambda^2) - \psi(\xi^2\zeta^2)} \\ &= \lim_{\xi \rightarrow \infty} \frac{\zeta^{2\alpha}\psi(\xi^2)}{\psi(\xi^2\zeta^2)} \frac{\lambda^2/(\xi\zeta)^2 - 1}{\psi(\lambda^2)/\psi(\xi^2\zeta^2) - 1} = 1 \end{aligned}$$

for all $\zeta > 0$. Hence, by dominated convergence,

$$\begin{aligned} \lim_{\xi \rightarrow \infty} \left(\frac{1}{\pi} \int_0^\infty \frac{\log \psi_\lambda(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta - \frac{1}{\pi} \int_0^\infty \frac{1}{1 + \zeta^2} \log \frac{\psi(\lambda^2)\zeta^{2-2\alpha}\xi^2}{\lambda^2\psi(\xi^2)} d\zeta \right) \\ = \lim_{\xi \rightarrow \infty} \int_0^\infty \frac{\log h(\xi, \zeta)}{1 + \zeta^2} d\zeta = 0. \end{aligned}$$

Since the integral of $(\log \zeta)/(1 + \zeta^2)$ over $(0, \infty)$ is zero, we have

$$\frac{1}{\pi} \int_0^\infty \frac{1}{1 + \zeta^2} \log \frac{\psi(\lambda^2)\zeta^{2-2\alpha}\xi^2}{\lambda^2\psi(\xi^2)} d\zeta = \frac{1}{2} \log \frac{\psi(\lambda^2)\xi^2}{\lambda^2\psi(\xi^2)}.$$

These two formulae yield that (see (3.1))

$$\psi_\lambda^\dagger(\xi) = \exp \left(\frac{1}{\pi} \int_0^\infty \frac{\log \psi_\lambda(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta \right) \sim \sqrt{\frac{\psi(\lambda^2)\xi^2}{\lambda^2\psi(\xi^2)}} \quad \text{as } \xi \rightarrow \infty.$$

Since $\mathcal{L}F_\lambda = c_\lambda \psi_\lambda^\dagger(\xi)/(\xi^2 + \lambda^2)$, this (and (4.8)) gives

$$\mathcal{L}F_\lambda(\xi) \sim \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} \frac{\lambda^2}{\xi^2} \sqrt{\frac{\psi(\lambda^2)\xi^2}{\lambda^2\psi(\xi^2)}} \quad \text{as } \xi \rightarrow \infty,$$

and (4.26) follows. Since F_λ is increasing on the interval $(0, \pi/2 - \vartheta_\lambda)$ (see (1.2)), formula (4.27) follows by Karamata's Tauberian theorem and monotone density theorem (see [6]). \square

5. SPECTRAL REPRESENTATION IN HALF-LINE

In this section we study the $L^2((0, \infty))$ properties of the operators $P_t^{(0, \infty)}$ and prove Theorems 1.3, 1.7 and 1.8. Let us define the operator Π^* by the formula

$$(5.1) \quad \Pi^* f(x) = \int_0^\infty f(\lambda) F_\lambda(x) d\lambda, \quad x > 0,$$

for $f \in C_c((0, \infty))$. Here $F_\lambda(x) = \sin(\lambda x + \vartheta_\lambda) - G_\lambda(x)$ is given by Theorem 1.1. At least formally, Π^* is the adjoint of Π defined in (1.5).

In this section, we write $\|f\|_2$ for $\|f\|_{L^2((0, \infty))}$, and $\langle f, g \rangle$ for $\langle f, g \rangle_{L^2((0, \infty))}$.

Lemma 5.1. *The operator Π^* extends to a bounded operator on $L^2((0, \infty))$.*

Proof. We follow the proof of Theorem 3 in [49]. Let $f \in C_c((0, \infty))$. Define

$$\Pi_1^* f(x) = \int_0^\infty f(\lambda) \sin(\lambda x + \vartheta_\lambda) d\lambda = \operatorname{Im} \mathcal{F} \tilde{f}(x), \quad x > 0,$$

where $\tilde{f}(\lambda) = e^{i\vartheta_\lambda} f(\lambda)$. Clearly, $\|\Pi_1^* f\|_2 \leq \|\mathcal{F} \tilde{f}\|_{L^2(\mathbf{R})} = \sqrt{2\pi} \|f\|_2$. Let

$$\Pi_2^* f(x) = \int_0^\infty f(\lambda) G_\lambda(x) d\lambda, \quad x > 0,$$

so that $\Pi^* = \Pi_1^* - \Pi_2^*$. We have

$$\int_0^\infty (\Pi_2^* f(x))^2 dx \leq \int_0^\infty \int_0^\infty \int_0^\infty |f(\lambda_1)| |f(\lambda_2)| G_{\lambda_1}(x) G_{\lambda_2}(x) dx d\lambda_1 d\lambda_2.$$

By Lemma 4.21, when $0 < \lambda_1 \leq \lambda_2$,

$$\begin{aligned} \int_0^\infty G_{\lambda_1}(x) G_{\lambda_2}(x) dx &\leq \sin \vartheta_{\lambda_1} \int_0^\infty G_{\lambda_2}(x) dx \\ &= \frac{\sin \vartheta_{\lambda_1}}{\lambda_2} \left(\cos \vartheta_{\lambda_2} - \sqrt{\frac{\lambda_2^2 \psi'(\lambda_2^2)}{\psi(\lambda_2^2)}} \right) \leq \frac{1}{\lambda_2}. \end{aligned}$$

By symmetry, for all $\lambda_1, \lambda_2 > 0$,

$$\int_0^\infty G_{\lambda_1}(x) G_{\lambda_2}(x) dx \leq \frac{1}{\max(\lambda_1, \lambda_2)}.$$

Hence, using Hardy-Hilbert's inequality (see e.g. [33]) in the last step,

$$\|\Pi_2^* f\|_2^2 = \int_0^\infty (\Pi_2^* f(x))^2 dx \leq \int_0^\infty \int_0^\infty \frac{|f(\lambda_1)| |f(\lambda_2)|}{\max(\lambda_1, \lambda_2)} d\lambda_1 d\lambda_2 < 4 \|f\|_2^2.$$

The lemma follows. \square

The extension of Π^* to $L^2((0, \infty))$ is denoted by the same symbol. By Lemma 5.1, we see that Π^* is indeed the adjoint of Π defined in (1.5).

Lemma 5.2. *We have $\langle \Pi^* f, \Pi^* g \rangle = (\pi/2) \langle f, g \rangle$ for $f, g \in L^2((0, \infty))$. Furthermore, for $f \in L^2((0, \infty))$ such that $e^{-t\psi(\lambda^2)} f(\lambda)$ is integrable in $\lambda > 0$, we have*

$$(5.2) \quad P_t^{(0, \infty)} \Pi^* f(x) = \int_0^\infty e^{-t\psi(\lambda^2)} f(\lambda) F_\lambda(x) d\lambda, \quad t, x > 0.$$

Proof. Again the argument follows the proof of Theorem 3 in [49]. Let $p_t^{(0, \infty)}(x, dy)$ be the kernel of $P_t^{(0, \infty)}$ (i.e. the \mathbf{P}_x distribution of the killed process $X_t^{(0, \infty)}$). If $f \in C_c((0, \infty))$, then $p_t^{(0, \infty)}(x, dy) f(\lambda) F_\lambda(y)$ is integrable in $y, \lambda > 0$. Theorem 1.1 and Fubini yield (5.2) in this case. The general case $f \in L^2((0, \infty))$ follows by approximation.

Let $f, g \in C_c((0, \infty))$, $k \in \mathbf{Z}$, and define $f_k(\lambda) = e^{-kt\psi(\lambda^2)}f(\lambda)$, $g_k(\lambda) = e^{-kt\psi(\lambda^2)}g(\lambda)$. From (5.2) it follows that $P_t^{(0, \infty)}\Pi^*f_k = \Pi^*f_{k+1}$ and $P_t^{(0, \infty)}\Pi^*g_k = \Pi^*g_{k+1}$. The operators $P_t^{(0, \infty)}$ are self-adjoint, so that

$$\begin{aligned} \langle \Pi^*f, \Pi^*g \rangle &= \left\langle P_t^{(0, \infty)}\Pi^*f_{-1}, \Pi^*g(x) \right\rangle \\ &= \left\langle \Pi^*f_{-1}, P_t^{(0, \infty)}\Pi^*g \right\rangle = \langle \Pi^*f_{-1}, \Pi^*g_1 \rangle. \end{aligned}$$

By induction, we have $\langle \Pi^*f, \Pi^*g \rangle = \langle \Pi^*f_{-k}, \Pi^*g_k \rangle$ for $k \geq 0$. In particular, if $\text{supp } f \subseteq (0, \lambda_0)$ and $\text{supp } g \subseteq (\lambda_0, \infty)$, then $\langle \Pi^*f, \Pi^*g \rangle = \langle e^{-kt\psi(\lambda_0^2)}f_{-k}, e^{kt\psi(\lambda_0^2)}g_k \rangle$. The right-hand side tends to zero as $k \rightarrow \infty$, so that Π^*f and Π^*g are orthogonal in $L^2((0, \infty))$. By approximation, this holds true for any $f, g \in L^2((0, \infty))$ such that $f(\lambda) = 0$ for $\lambda \geq \lambda_0$ and $g(\lambda) = 0$ for $\lambda \leq \lambda_0$.

Define $m(E) = \|\Pi^*\mathbf{1}_E\|_2^2$ for Borel $E \subseteq (0, \infty)$. Clearly, $0 \leq m(E) \leq C\|\mathbf{1}_E\|_2^2 = C|E|$. If $E_1 \subseteq (0, \lambda_0)$ and $E_2 \subseteq (\lambda_0, \infty)$, then

$$m(E_1 \cup E_2) = \|\Pi^*\mathbf{1}_{E_1}\|_2^2 + \|\Pi^*\mathbf{1}_{E_2}\|_2^2 + 2\langle \Pi^*\mathbf{1}_{E_1}, \Pi^*\mathbf{1}_{E_2} \rangle = m(E_1) + m(E_2).$$

Finally, suppose that $E = \bigcup_{n=1}^\infty E_n$, where $E_1 \subseteq E_2 \subseteq \dots$ and $|E| < \infty$. Since $\mathbf{1}_{E_n}$ converges to $\mathbf{1}_E$ in $L^2((0, \infty))$, also $\Pi^*\mathbf{1}_{E_n}$ converges to $\Pi^*\mathbf{1}_E$ in $L^2((0, \infty))$, and so $m(E) = \lim_{n \rightarrow \infty} m(E_n)$. It follows that m is an absolutely continuous measure on $(0, \infty)$, and by approximation,

$$\langle \Pi^*f, \Pi^*g \rangle = \int_0^\infty f(\lambda)g(\lambda)m(d\lambda)$$

for any $f, g \in L^2((0, \infty))$. The lemma will be proved if we show that $m(E) = (\pi/2)|E|$.

Fix $\lambda_0 > 0$ and define $f_\delta = (1/\sqrt{\delta})\mathbf{1}_{[\lambda_0, \lambda_0+\delta]}$ for $\delta > 0$. Clearly, $\|f_\delta\|_2 = 1$, and $m([\lambda_0, \lambda_0 + \delta])/\delta = \|\Pi^*f_\delta\|_2^2$. Furthermore, $\sqrt{\delta}\Pi^*f_\delta = g_1 + g_2 - g_3$, where

$$\begin{aligned} g_1(x) &= \int_{\lambda_0}^{\lambda_0+\delta} \sin(\lambda x + \theta_{\lambda_0})d\lambda, \\ g_2(x) &= \int_{\lambda_0}^{\lambda_0+\delta} (\sin(\lambda x + \theta_\lambda) - \sin(\lambda x + \theta_{\lambda_0}))d\lambda, \\ g_3(x) &= \int_{\lambda_0}^{\lambda_0+\delta} G_\lambda(x)d\lambda. \end{aligned}$$

We study the behavior of g_1 , g_2 and g_3 as $\delta \rightarrow 0^+$.

By Lemma 4.21, we have $G_\lambda(x) \leq 1$ and $\int_0^\infty G_\lambda(x)dx \leq 1/\lambda$. Hence, by Fubini,

$$\|g_3\|_2^2 = \int_0^\infty \left(\int_{\lambda_0}^{\lambda_0+\delta} G_\lambda(x)dx \right)^2 dx \leq \int_0^\infty \delta \left(\int_{\lambda_0}^{\lambda_0+\delta} G_\lambda(x)dx \right) dx \leq \frac{\delta^2}{\lambda_0},$$

so that $\|g_3\|_2/\sqrt{\delta}$ tends to zero as $\delta \rightarrow 0^+$.

The function g_2 is the imaginary part of the Fourier transform of the function $\tilde{f}(\lambda) = (\exp(i\vartheta_\lambda) - \exp(i\vartheta_{\lambda_0}))\mathbf{1}_{[\lambda_0, \lambda_0+\delta]}$. By Proposition 4.17, we have

$|\vartheta_\lambda - \vartheta_{\lambda_0}| \leq C|\lambda - \lambda_0|$ for some $C > 0$ (depending on λ_0). Hence $|\tilde{f}(\lambda)| \leq C\delta$, and so $\|\tilde{f}\|_2 \leq C\delta^{3/2}$. It follows that $\|g_2\|_2/\sqrt{\delta} \leq \sqrt{2\pi} \|\tilde{f}\|_2/\sqrt{\delta} \leq \sqrt{2\pi} C\delta$ also tends to zero as $\delta \rightarrow 0^+$.

Finally, for the function g_1 , by Fubini,

$$\begin{aligned} & 2 \int_0^\infty (g_1(x))^2 e^{-\varepsilon x} dx \\ &= \int_{\lambda_0}^{\lambda_0+\delta} \int_{\lambda_0}^{\lambda_0+\delta} \int_0^\infty 2 \sin(\lambda_1 x + \theta_{\lambda_0}) \sin(\lambda_2 x + \theta_{\lambda_0}) e^{-\varepsilon x} dx d\lambda_1 d\lambda_2 \\ &= \int_{\lambda_0}^{\lambda_0+\delta} \int_{\lambda_0}^{\lambda_0+\delta} \int_0^\infty (\cos((\lambda_1 - \lambda_2)x) - \cos((\lambda_1 + \lambda_2)x + 2\theta_{\lambda_0})) e^{-\varepsilon x} dx d\lambda_1 d\lambda_2 \\ &= \int_{\lambda_0}^{\lambda_0+\delta} \int_{\lambda_0}^{\lambda_0+\delta} \left(\frac{\varepsilon}{\varepsilon^2 + (\lambda_1 - \lambda_2)^2} + \frac{\varepsilon \cos(2\vartheta_{\lambda_0}) - (\lambda_1 + \lambda_2) \sin(2\vartheta_{\lambda_0})}{\varepsilon^2 + (\lambda_1 + \lambda_2)^2} \right) d\lambda_1 d\lambda_2. \end{aligned}$$

By taking the limit $\varepsilon \rightarrow 0^+$ and using dominated convergence for the second term of the integrand, we obtain that

$$\|g_1\|_2^2 = \int_0^\infty (g_1(x))^2 dx = \frac{1}{2} \int_{\lambda_0}^{\lambda_0+\delta} \left(\pi - \sin(2\vartheta_{\lambda_0}) \log \frac{\lambda_0 + \delta + \lambda_2}{\lambda_0 + \lambda_2} \right) d\lambda_2.$$

Hence, using $0 \leq \log(1 + s) \leq s$ ($s \geq 0$),

$$\lim_{\delta \rightarrow 0^+} \frac{\|g_1\|_2^2}{\delta} = \frac{\pi}{2} - \lim_{\delta \rightarrow 0^+} \frac{\sin(2\vartheta_{\lambda_0})}{2\delta} \int_{\lambda_0}^{\lambda_0+\delta} \log \left(1 + \frac{\delta}{\lambda_0 + \lambda_2} \right) d\lambda_2 = \frac{\pi}{2}.$$

We conclude that for all $\lambda_0 > 0$,

$$\lim_{\delta \rightarrow 0^+} \frac{m([\lambda_0, \lambda_0 + \delta])}{\delta} = \lim_{\delta \rightarrow 0^+} \left(\frac{\|g_1 + g_2 - g_3\|_2}{\sqrt{\delta}} \right)^2 = \frac{\pi}{2},$$

that is, $m(d\lambda) = (\pi/2)d\lambda$. \square

Proof of Theorem 1.3 when Π is injective. By Lemma 5.2, $\sqrt{2/\pi} \Pi^*$ is an $L^2((0, \infty))$ isometry, and by the extra assumption, its adjoint $\sqrt{2/\pi} \Pi$ is injective. Hence, both operators are unitary on $L^2((0, \infty))$. The theorem follows now directly from Lemma 5.2 and $(\Pi^*)^{-1} = (2/\pi)\Pi^*$. \square

In [56], Theorem 1.3 is proved in full generality. More precisely, it is shown that Π is always injective. Here we are satisfied with the following observation: When X_t is a symmetric α -stable process, then, by Example 6.1 below, $F_\lambda(x) = F_x(\lambda)$, and therefore $\Pi^* = \Pi$. Since a self-adjoint isometry is necessarily unitary, we see (by Lemma 5.2) that Π is injective. Hence, we have proved the following result.

Corollary 5.3. *When $\alpha \in (0, 2]$ and X_t is the symmetric α -stable process (that is, $\psi(\xi) = \xi^{\alpha/2}$), then the hypothesis of Theorem 1.3 holds true. \square*

In the following proofs of Theorems 1.7 and 1.8, we use the full statement of Theorem 1.3. In other words, we assume that Π is injective (which is proved to be true in general in [56]).

Proof of Theorem 1.7. Let $f \in C_c((0, \infty))$. By Theorem 1.3,

$$P_t^{(0, \infty)} f(x) = \frac{2}{\pi} \Pi^*(e^{-t\psi(\lambda^2)} \Pi f(\lambda))(x) = \frac{2}{\pi} \int_0^\infty e^{-t\psi(\lambda^2)} F_\lambda(x) \Pi^* f(\lambda) d\lambda$$

for all $t, x > 0$. The function $e^{-t\psi(\lambda^2)} F_\lambda(x) F_\lambda(y) f(y)$ is jointly integrable in $y, \lambda > 0$. By Fubini,

$$P_t^{(0, \infty)} f(x) = \int_0^\infty \left(\frac{2}{\pi} \int_0^\infty e^{-t\psi(\lambda^2)} F_\lambda(x) F_\lambda(y) d\lambda \right) f(y) dy.$$

Since $f \in C_c((0, \infty))$ is arbitrary, the kernel of $P_t^{(0, \infty)}$ has the form (1.6). \square

Proof of Theorem 1.8 under additional assumption (1.11). By monotone convergence,

$$\mathbf{P}_x(\tau_{(0, \infty)} > t) = P_t^{(0, \infty)} \mathbf{1}_{(0, \infty)}(x) = \lim_{\varepsilon \rightarrow 0^+} P_t^{(0, \infty)} e_\varepsilon(x),$$

where $e_\varepsilon(x) = e^{-\varepsilon x} \mathbf{1}_{(0, \infty)}(x)$. We have

$$\Pi e_\varepsilon(\lambda) = \int_0^\infty e^{-\varepsilon x} F_\lambda(x) dx = \mathcal{L}F_\lambda(\varepsilon).$$

By Theorem 1.3,

$$(5.3) \quad P_t^{(0, \infty)} e_\varepsilon(x) = \frac{2}{\pi} \int_0^\infty e^{-t\psi(\lambda^2)} F_\lambda(x) \mathcal{L}F_\lambda(\varepsilon) d\lambda.$$

If we can use dominated convergence, then, by Proposition 4.20,

$$\begin{aligned} \mathbf{P}_x(\tau_{(0, \infty)} > t) &= \frac{2}{\pi} \int_0^\infty e^{-t\psi(\lambda^2)} F_\lambda(x) \left(\lim_{\varepsilon \rightarrow 0^+} \mathcal{L}F_\lambda(\varepsilon) \right) d\lambda \\ &= \frac{2}{\pi} \int_0^\infty e^{-t\psi(\lambda^2)} F_\lambda(x) \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} d\lambda, \end{aligned}$$

as desired. Hence, it remains to show that the integrand in (5.3) is dominated by an integrable function when $\varepsilon \rightarrow 0^+$.

By the assumption (1.11), there are $C, \alpha > 0$ such that $\psi(\xi) \geq C\xi^\alpha$ for $\xi > 1$ (see the proof of Corollary 4.25). Hence, $e^{-t\psi(\lambda^2)}$ is integrable in $\lambda > 0$. By Proposition 4.22, we have $\mathcal{L}F_\lambda(\varepsilon) \leq 2/\lambda$, and by Corollary 4.25 (for $x_1 = 0$ and $x_2 = x$), $|F_\lambda(x)| \leq C \min(1, \lambda^{\alpha/2})$ for some $C > 0$ (depending on x). Hence, the integrand in (5.3) is bounded by $2Ce^{-t\psi(\lambda^2)} \min(1, \lambda^{\alpha/2})/\lambda$, which is integrable in $\lambda > 0$. Formula (1.9) is proved.

By the same argument, the integrand in (1.10) is integrable jointly in $\lambda > 0$ and $t > t_0$. Formula (1.10) follows by integrating both sides of it in t over (t_0, ∞) , using Fubini and comparing the result with (1.9). \square

6. EXAMPLES

Example 6.1. Let $\psi(\xi) = \xi^{\alpha/2}$, where $\alpha \in (0, 2)$. Then Z_t is the $(\alpha/2)$ -stable subordinator, and $X_t = B_{Z_t}$ is the symmetric α -stable Lévy process. By (4.17) and (4.16),

$$\vartheta_\lambda = \frac{1}{\pi} \int_0^1 \frac{1}{1-z^2} \log \frac{1-z^\alpha}{(z^{-\alpha}-1)z^2} dz = \frac{2-\alpha}{\pi} \int_0^1 \frac{-\log z}{1-z^2} dz = \frac{(2-\alpha)\pi}{8}.$$

(Note that the *phase-shift* of $(2-\alpha)\pi/8$ was conjectured, in somewhat different setting, in [73]; see [54] for further discussion.) By Theorem 1.1, the eigenfunctions of $P_t^{(0,\infty)}$ and $\mathcal{A}_{(0,\infty)}$ are given by the formula

$$F_\lambda(x) = \sin(\lambda x + (2-\alpha)\pi/8) - \int_0^\infty \gamma_\lambda(\xi) e^{-x\xi} d\xi, \quad x > 0,$$

where (see (4.15) and (3.1))

$$\begin{aligned} \gamma_\lambda(\xi) &= \frac{\sqrt{2\alpha} \lambda^{\alpha-1}}{2\pi} \frac{\xi^\alpha \sin(\alpha\pi/2)}{\lambda^{2\alpha} + \xi^{2\alpha} - 2\lambda^\alpha \xi^\alpha \cos(\alpha\pi/2)} \times \\ &\quad \times \exp\left(\frac{1}{\pi} \int_0^\infty \frac{1}{1+\zeta^2} \log \frac{1-\xi^2\zeta^2/\lambda^2}{1-\xi^\alpha\zeta^\alpha/\lambda^\alpha} d\zeta\right). \end{aligned}$$

Clearly $\gamma(s) = \lambda\gamma_\lambda(\lambda s)$ does not depend on λ , and finally,

$$F_\lambda(x) = F(\lambda x) = \sin(\lambda x + (2-\alpha)\pi/8) - \int_0^\infty \gamma(s) e^{-\lambda s x} ds,$$

where

$$\begin{aligned} \gamma(s) &= \frac{\sqrt{2\alpha} \sin(\alpha\pi/2)}{2\pi} \frac{s^\alpha}{1+s^{2\alpha} - 2s^\alpha \cos(\alpha\pi/2)} \\ &\quad \times \exp\left(\frac{1}{\pi} \int_0^\infty \frac{1}{1+\zeta^2} \log \frac{1-s^2\zeta^2}{1-s^\alpha\zeta^\alpha} d\zeta\right). \end{aligned}$$

By Lemma 4.27, we have $F_\lambda(x) \sim (\sqrt{\alpha/2} \Gamma(\alpha/2))^{-1} (\lambda x)^{\alpha/2}$ as $x \rightarrow 0^+$. By Theorem 1.3 (or Corollary 5.3), the functions F_λ yield a generalized eigenfunction expansion of $\mathcal{A}_{(0,\infty)}$ and $P_t^{(0,\infty)}$, and by Theorems 1.7 and 1.8,

$$\begin{aligned} p_t^{(0,\infty)}(x, y) &= \frac{2}{\pi} \int_0^\infty e^{-t\lambda^\alpha} F(\lambda x) F(\lambda y) d\lambda, \quad t, x, y > 0, \\ \mathbf{P}_x(\tau_{(0,\infty)} > t) &= \frac{\sqrt{2\alpha}}{\pi} \int_0^\infty \frac{e^{-t\lambda^\alpha} F(\lambda x)}{\lambda} d\lambda, \quad t, x > 0. \end{aligned}$$

For $\alpha = 1$, these results were obtained in [49], and the formulae for $p_t^{(0,\infty)}(x, y)$ and $\mathbf{P}_x(\tau_{(0,\infty)} > t)$ can be significantly simplified in this case.

Example 6.2. If $\psi(\xi) = \sqrt{m^2 + \xi} - m$ for some $m > 0$, then Z_t is the relativistic subordinator with mass m , and the corresponding Lévy process is the relativistic 1-stable process with mass m . Its infinitesimal operator \mathcal{A} is the (quasi-)relativistic Hamiltonian of a free particle with mass m , and

$\mathcal{A}_{(0,\infty)}$ is the Hamiltonian corresponding to the semi-infinite potential well. By (4.17),

$$\vartheta_\lambda = \frac{1}{\pi} \int_0^1 \frac{1}{1-z^2} \log \frac{\sqrt{m^2 + \lambda^2} - \sqrt{m^2 + \lambda^2 z^2}}{\left(\sqrt{m^2 + \lambda^2/z^2} - \sqrt{m^2 + \lambda^2}\right) z^2} dz,$$

which increases with λ from 0 as $\lambda \rightarrow 0^+$ to $\pi/8$ as $\lambda \rightarrow \infty$. The eigenfunctions of $P_t^{(0,\infty)}$ have the form $F_\lambda(x) = \sin(\lambda x + \vartheta_\lambda) - G_\lambda(x)$, where G_λ is the Laplace transform of an explicit nonnegative function. By Lemma 4.27, $F_\lambda(x) \sim \sqrt{2\lambda x/\pi}$ as $x \rightarrow 0^+$. See [38] for a detailed discussion of this example.

Example 6.3. Let $\psi(\xi) = \xi^{\alpha/2} + \beta\xi$, $\alpha \in (0, 2)$, $\beta > 0$. Then Z_t is the $(\alpha/2)$ -stable subordinator with drift, and the corresponding Lévy process X_t is a mixture of the Brownian motion and the symmetric α -stable Lévy process. When $\alpha = 1$, then, by (4.17),

$$\begin{aligned} \vartheta_\lambda &= \frac{1}{\pi} \int_0^1 \frac{1}{1-\zeta^2} \log \frac{(\lambda + \beta\lambda^2) - (\lambda\zeta + \beta\lambda^2\zeta^2)}{\left(\left(\frac{\lambda}{\zeta} + \frac{\beta\lambda^2}{\zeta^2}\right) - (\lambda + \beta\lambda^2)\right)\zeta^2} d\zeta \\ &= \frac{1}{\pi} \int_0^1 \frac{1}{1-\zeta^2} \log \frac{1 + \beta\lambda + \beta\lambda\zeta}{\zeta + \beta\lambda + \beta\lambda\zeta} d\zeta. \end{aligned}$$

This decreases with λ from $\pi/8$ as $\lambda \rightarrow 0^+$ to 0 as $\lambda \rightarrow \infty$, and can be written explicitly in terms of the dilogarithm function Li_2 . For general α , the expression for ϑ_λ is more complicated, and it can be proved that ϑ_λ decreases from $(2 - \alpha)\pi/8$ to 0. By Lemma 4.27, $F_\lambda(x) \sim \sqrt{\beta + (\alpha/2)\lambda^{\alpha-2}} \lambda x$ as $x \rightarrow 0^+$.

Example 6.4. Let $\psi(\xi) = \log(1 + \xi)$, so that Z_t is the gamma subordinator. The subordinate process X_t is called variance gamma process. By (4.17),

$$\vartheta_\lambda = \frac{1}{\pi} \int_0^1 \frac{1}{1-z^2} \log \frac{\log(1 + \lambda^2) - \log(1 + \lambda^2 z^2)}{(\log(1 + \lambda^2/z^2) - \log(1 + \lambda^2)) z^2} dz.$$

It can be proved that ϑ_λ increases with λ from 0 as $\lambda \rightarrow 0^+$ to $\pi/4$ as $\lambda \rightarrow \infty$. By Lemma 4.27, the eigenfunctions F_λ satisfy $F_\lambda(x) \sim C_\lambda/\sqrt{|\log x|}$ as $x \rightarrow 0^+$, with $C_\lambda = \lambda/\sqrt{2 + 2\lambda^2}$. In particular, F_λ is not Hölder continuous. Note that the assumptions of Theorem 1.7 are satisfied only for $t > 1/2$, but Theorem 1.8 applies for all $t > 0$. On the other hand, the more restrictive condition (1.11) is not met, so this case is not covered by the proof of Theorem 1.8 given above. The general argument of [56] needs to be used.

Example 6.5. Let $\psi(\xi) = \log(1 + \log(1 + \xi))$. It can be verified that in this case ϑ_λ is greater than $\pi/4$ for some λ , e.g. $\vartheta_8 \approx 0.287\pi$. This proves that it is not true in general that $\vartheta_\lambda \leq \pi/4$, even if ψ is unbounded. Furthermore, $e^{-t\psi(\lambda^2)}$ is not integrable in $\lambda > 0$ for any $t > 0$, so Theorem 1.7 is of no use here. Theorem 1.8 applies for $t > 1/2$, but the additional assumption (1.8) is not satisfied.

Example 6.6. Let $\psi(\xi) = \xi/(1 + \xi)$. In this case the subordinator Z_t is the compound Poisson process with exponential jump distribution, and the jumps of X_t have Laplace distribution with density $(1/2)e^{-|x|}$. Note that $1/(\psi(\lambda^2) - \psi(\zeta))$ is holomorphic in $\zeta \in \mathbf{C} \setminus \{\lambda\}$, and hence γ_λ vanishes. It follows that $F_\lambda = \sin(\lambda x + \vartheta_\lambda)\mathbf{1}_{(0,\infty)}(x)$. By (1.3) and a contour integration,

$$\vartheta_\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2 - z^2} \log \frac{1 + z^2}{1 + \lambda^2} dz = \arctan \lambda;$$

we omit the details. Hence, $F_\lambda(x) = \sin(\lambda x + \arctan \lambda)\mathbf{1}_{(0,\infty)}(x)$ satisfies $\mathcal{A}F_\lambda(x) = -\psi(\lambda^2)F_\lambda(x)$ for all $x \in (0, \infty)$. This can be alternatively proved by a direct calculation.

It can be easily proved that γ_λ vanishes if and only if $\psi(\xi) = \beta\xi$ (i.e. X_t is a Brownian motion) or $\psi(\xi) = C_1\xi/(\xi + C_2)$ for some $C_1, C_2 > 0$ (which corresponds to the process X_t studied in this example, up to time and space scaling).

Example 6.7. In general, γ_λ may have non-zero singular part. For example, consider $\psi(\xi) = 5x/(x + 1) + x/(x + 5)$ and $\lambda = 1$. Then $\psi^+(-4) = 8/3 = \psi(1)$, and hence γ_λ has an atom at 2 for $\lambda = 1$.

7. APPLICATION TO SYSTEMS OF PDES

As it was the case in [49], where the Cauchy process was studied, the eigenfunctions F_λ are the boundary values of solutions of a certain system of PDEs, a spectral problem with spectral parameter in the boundary. In [49], the formula for F_λ (for the case of the Cauchy process) was found by solving the corresponding system of PDEs. Here the argument goes in the opposite direction, and the result for PDEs is an application of the formula for F_λ .

We need some background on Kreĭn correspondence. For a complete Bernstein function ψ , there is a unique *string* m (a non-decreasing, right-continuous unbounded function from $[0, \infty)$ to $[0, \infty]$) which is the *Kreĭn representation* of ψ . For simplicity, we only consider the case when $m(0) = 0$, m is continuously differentiable on $(0, \infty)$, and $m'(y) > 0$ for all $y > 0$. Denote $a(y) = 1/m'(y)$. Then, for each $\lambda \geq 0$ there exists a (unique) nonnegative, non-increasing continuous function $g_\lambda(y) = g(\lambda, y)$ such that $a(y)g_\lambda''(y) = \lambda g_\lambda(y)$ for $y > 0$, and $g_\lambda(0) = 1$. Furthermore,

$$\psi(\lambda) = -g_\lambda'(0), \quad \lambda \geq 0.$$

For the details, we refer to [45], or Chapter 14 in [66], and the references therein. The Kreĭn's correspondence between m and ψ is a rather complicated concept, and there are only few examples of explicit pairs, see [66].

Let $f \in \mathcal{S}$ and define $u(x, y)$ using the Fourier transform in x ,

$$\mathcal{F}_x u(\xi, y) = g(\xi^2, y) \mathcal{F}f(\xi), \quad \xi \in \mathbf{R}, y \geq 0.$$

It is easy to check that u is the solution of the problem

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + a(y) \frac{\partial^2}{\partial y^2} \right) u(x, y) &= 0 & x \in \mathbf{R}, y > 0; \\ u(x, 0) &= f(x), & x \in \mathbf{R}. \end{aligned}$$

Furthermore,

$$\mathcal{F}_x \left(\frac{\partial}{\partial y} u \right) (\xi, 0) = \lim_{y \rightarrow 0^+} \frac{g(\xi^2, y) - 1}{y} \mathcal{F}f(\xi) = -\psi(\xi^2) \mathcal{F}f(\xi),$$

so that $(\partial/\partial y)u(x, 0) = \mathcal{A}f(x)$, where \mathcal{A} is the operator with Fourier symbol $-\psi(\xi^2)$. Theorems 1.1 implies the following result.

Theorem 7.1. *Let $a(y)$ be a positive continuous function on $(0, \infty)$ such that $1/a$ is integrable over $(0, 1)$, but not over $(1, \infty)$. For $\lambda \geq 0$, let $g_\lambda(y) = g(\lambda, y)$ be the nonnegative, non-increasing continuous solution of $a(y)g''_\lambda(y) = \lambda g_\lambda(y)$ ($y > 0$), satisfying $g_\lambda(0) = 1$. Define $\psi(\lambda) = g'_\lambda(0)$, and let F_λ be the function given in Theorem 1.1. Finally, let*

$$(7.1) \quad u_\lambda(x, y) = \mathcal{F}_x^{-1}(g(\xi^2, y) \mathcal{F}F_\lambda(\xi)), \quad x \in \mathbf{R}, y \geq 0.$$

Then u_λ is continuous and bounded in $x \in \mathbf{R}, y \geq 0$, twice differentiable in $x \in \mathbf{R}, y > 0$, and it satisfies

$$(7.2) \quad \begin{aligned} \left(\frac{\partial^2}{\partial x^2} + a(y) \frac{\partial^2}{\partial y^2} \right) u_\lambda(x, y) &= 0 & x \in \mathbf{R}, y > 0; \\ u_\lambda(x, 0) &= 0, & x < 0; \\ \frac{d}{dy} u_\lambda(x, 0) &= -\psi(\lambda^2) u_\lambda(x, 0), & x > 0. \end{aligned}$$

The Fourier transform and inverse Fourier transform in (7.1) needs to be understood in the distributional sense.

Sketch of the proof. Using the properties of $g(\lambda, y)$, it is easy to check that u_λ is a weak solution of (7.2). Since $g(\xi^2, y) \mathcal{F}F_\lambda(\xi)$ (for a fixed $y \geq 0$) is a sum of an $L^2(\mathbf{R})$ function and $C_y/(\xi - \lambda) - \bar{C}_y/(\xi + \lambda)$ (in a similar way as in Section 4), $u(x, y)$ is a bounded function. Continuity of u_λ and its first and second partial derivatives in $x \in \mathbf{R}, y > 0$ follows by elliptic regularity theorem. Continuity of u_λ on the boundary is proved using the properties of $g(\lambda, y)$ and $F_\lambda(x)$. \square

We provide two examples. If $a(y) = 1/(1+2ay)$ for a constant $a > 0$, then we find that $g_\lambda(y) = (1+2ay)^{-(\sqrt{\lambda+a^2}-a)/(2a)}$ and $\psi(\lambda) = \sqrt{\lambda+a^2}-a$. Hence ψ is the characteristic exponent of the relativistic subordinator, described in Example 6.2. For a related example, see Section 2.7 in [60].

Consider now $\alpha \in (0, 2)$ and $a(y) = \alpha^2 c_\alpha^2 y^{2-2/\alpha}$, where $c_\alpha = 2^{-\alpha} \Gamma(1 - \alpha/2) / \Gamma(1 + \alpha/2)$. We have $g_\lambda(y) = C_\alpha (c_\alpha \sqrt{\lambda} y)^{1/2} K_{\alpha/2}((c_\alpha \sqrt{\lambda} y)^{1/\alpha})$, where $K_{\alpha/2}$ is the modified Bessel function of the second kind and $C_\alpha = \alpha 2^{-\alpha/2} / \Gamma(1 + \alpha/2)$. Moreover, $\psi(\lambda) = \lambda^{\alpha/2}$, so that ψ is the characteristic exponent of the $(\alpha/2)$ -stable subordinator, considered in Example 6.1. When $\alpha = 1$, $a(y) = 1$.

The problem (7.2) with $a(y) = 1$ was studied in [49] to find the eigenfunctions of $P_t^{(0, \infty)}$. Earlier, a similar relation for more general domains (also in higher dimensions) was applied e.g. in [2, 3, 10, 70], and for general

symmetric stable processes e.g. in [19, 20, 21, 59]. Related problems appear frequently in hydrodynamics (the *sloshing problem*), see the references in [49].

The operator $d^2/dx^2 + a(y)d^2/dy^2$ is the generator of a two-dimensional diffusion Y_t on $\mathbf{R} \times [0, \infty)$, with reflecting boundary. By repeating the construction given in Example 3.1 in [42], we can identify the subordinate Brownian motion X_t studied in the present article with the trace left on the horizontal axis by Y_t . We remark that the connection between subordinate Brownian motion in \mathbf{R}^d and traces of diffusions in \mathbf{R}^{d+1} was applied e.g. in [12, 13] to find formulae for the distribution of some Lévy processes stopped at the time of first exit from the half-line or the interval. A related problem for the trace of a two-dimensional jump-type stable process was studied in [36].

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REFERENCES

- [1] D. Applebaum, *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge, 2004.
- [2] R. Bañuelos, T. Kulczycki, *The Cauchy process and the Steklov problem*. J. Funct. Anal. 211(2) (2004), pp. 355–423.
- [3] R. Bañuelos, T. Kulczycki, *Spectral gap for the Cauchy process on convex symmetric domains*. Comm. Partial Diff. Equations 31 (2006), pp. 1841–1878.
- [4] G. Baxter, M. D. Donsker, *On the distribution of the supremum functional for processes with stationary independent increments*. Trans. Amer. Math. Soc. 85 (1957) 73–87.
- [5] J. Bertoin, *Lévy Processes*. Cambridge University Press, Melbourne–New York, 1998.
- [6] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular Variation*. Cambridge University Press, Cambridge, 1987.
- [7] R. M. Blumenthal, R. K. Gettoor, *Markov Processes and Potential Theory*. Academic Press, Reading, MA, 1968.
- [8] K. Bogdan, T. Byczkowski, *Potential theory for the α -stable Schrödinger operator on bounded Lipschitz domains*. Studia Math. 133(1) (1999) 53–92.

- [9] K. Bogdan, T. Byczkowski, *Potential theory of Schrödinger operator based on fractional Laplacian*. Probab. Math. Statist. 20(2) (2000), pp. 293–335.
- [10] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song, Z. Vondracek, *Potential Analysis of Stable Processes and its Extensions*. Lecture Notes in Mathematics 1980, Springer, 2009.
- [11] K. Bogdan, T. Grzywny, M. Ryznar, *Heat kernel estimates for the fractional Laplacian*. Annals Probab. 38(5) (2010), pp. 1901–1923.
- [12] T. Byczkowski, J. Małecki, M. Ryznar, *Bessel potentials, hitting distributions and Green functions*. Trans. Amer. Math. Soc. 361 (2009), pp. 4871–4900.
- [13] T. Byczkowski, J. Małecki, M. Ryznar, *Hitting half-spaces by Bessel-Brownian diffusions*. Potential Anal. Volume 33(1) (2010), pp. 47–83.
- [14] Z. Q. Chen, P. Kim, R. Song, *Sharp Heat Kernel Estimates for Relativistic Stable Processes in Open Sets*. Preprint (2009) arXiv:0908.1509v1, To appear in Ann. Probab.
- [15] Z. Q. Chen, R. Song, *Intrinsic ultracontractivity and conditional gauge for symmetric stable processes*. J. Funct. Anal. 150(1) (1997) 204–239.
- [16] Z.Q. Chen, R. Song, *Two sided eigenvalue estimates for subordinate processes in domains*. J. Funct. Anal. 226(1) (2005), pp. 90–113.
- [17] Z.Q. Chen, R. Song, *Spectral properties of subordinate processes in domains*. Stochastic Analysis and Partial Differential Equations, pp. 77–84 (Eds. G-Q Chen, E. Hsu and M. Pinsky), AMS Contemp. Math. 429 (2007).
- [18] K. L. Chung, *Doubly-Feller process with multiplicative functional*. In: *Seminar on Stochastic Processes, 1985* (Gainesville, Fla., 1985), Progr. Probab. Statist. 12, Birkhäuser, Boston, Boston, MA., 1986, 63–78.
- [19] R. D. DeBlassie, *The first exit time of a two-dimensional symmetric stable process from a wedge*. Ann. Probab. 18 (1990), pp. 1034–1070.
- [20] R. D. DeBlassie, *Higher order PDE's and symmetric stable processes*. Probab. Theory Related Fields 129 (2004), pp. 495–536.
- [21] R. D. DeBlassie, P. J. Méndez-Hernández, *α -continuity properties of the symmetric α -stable process*. Trans. Amer. Math. Soc. 359 (2007), pp. 2343–2359.
- [22] P. Dierolf, J. Voigt, *Convolution and \mathcal{S}' -convolution of distributions*. Collectanea Math. 29 (1978), pp. 185–196.
- [23] R. A. Doney, *Fluctuation Theory for Lévy Processes*. Lecture Notes in Math. 1897, Springer, Berlin, 2007.

- [24] R. A. Doney, M. S. Savov, *The asymptotic behavior of densities related to the supremum of a stable process*. Ann. Probab. 38(1) (2010), pp. 316–326.
- [25] E. B. Dynkin, *Markov processes, Vols. I and II*. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1965.
- [26] M. J. Esteban, M. Lewin, E. Séré, *Variational methods in relativistic quantum mechanics*. Bull. Amer. Math. Soc. 45(4) (2008) pp. 545–593.
- [27] R. L. Frank, L. Geisinger, *Refined semiclassical asymptotics for fractional powers of the Laplace operator*. Preprint 2011, arXiv:1105.5181v1.
- [28] B. E. Fristedt, *Sample functions of stochastic processes with stationary, independent increments*. In: *Advances in Probability and Related Topics*, vol. 3, Dekker, New York, 1974, 241–396.
- [29] P. Garbaczewski, J. R. Klauder, R. Olkiewicz, *Schrödinger problem, Lévy processes, and noise in relativistic quantum mechanics*. Phys. Rev. E 51 (1995), pp. 4114–4131.
- [30] R. K. Gettoor, *Markov operators and their associated semi-groups*. Pacific J. Math. 9 (1959), pp. 449–472.
- [31] I. C. Gohberg, S. Goldberg, M. A. Kaashoek, *Classes of linear operators. Vol. 1*. Birkhäuser, 1990.
- [32] P. Graczyk, T. Jakubowski, *On Wiener-Hopf factors of stable processes*. Ann. Inst. Henri Poincaré (B) 47(1) (2010), pp. 9–19.
- [33] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*. Cambridge Univ. Press, London, 1952.
- [34] Y. Hirata, H. Ogata, *On the exchange formula for distributions*. J. Sci. Hiroshima Univ. Ser. A 22 (1958), pp. 147–152.
- [35] T. R. Hurd, A. Kuznetsov, *On the first passage time for Brownian motion subordinated by a Levy process*. J. Appl. Prob. 46 (2009), pp. 181–198.
- [36] Y. Isozaki, *Hitting of a line or a half-line in the plane by two-dimensional symmetric stable Lévy processes*. Stoch. Proc. Appl. 121(8) (2011), pp. 1749–1769.
- [37] N. Jacob, *Pseudo Differential Operators and Markov Processes*, Vol. 1. Imperial College Press, London, 2001
- [38] K. Kaleta, M. Kwaśnicki, J. Małecki, *One-dimensional quasi-relativistic particle in the box*. In preparation.
- [39] A. Kamiński, *Convolution, product and Fourier transform of distributions*. Studia Math. 74 (1982), pp. 83–96.

- [40] P. Kim, R. Song, Z. Vondraček, *Boundary Harnack Principle for Subordinate Brownian Motions*. Stoch. Proc. Appl. 119(5) (2009), pp. 1601–1631.
- [41] P. Kim, R. Song, Z. Vondraček, *On the potential theory of one-dimensional subordinate Brownian motions with continuous components*. Potential Anal. 33(2) (2010), pp. 153–173
- [42] P. Kim, R. Song, Z. Vondraček, *On harmonic functions for trace processes*. Math. Nachrichten (2010), to appear.
- [43] P. Kim, R. Song, Z. Vondraček, *Two-sided Green function estimates for killed subordinate Brownian motions*. Preprint, 2010, arXiv:1007.5455v2.
- [44] P. Kim, R. Song, Z. Vondraček, *Potential theory of subordinate Brownian motions revisited*. Preprint, 2011, arXiv:1102.1369v2.
- [45] S. Kotani, S. Watanabe, *Krein's spectral theory of strings and generalized diffusion processes*. In *Functional analysis in Markov processes (Katata/Kyoto, 1981)*, pp. 235–259, Lecture Notes in Math. 923, Springer, Berlin, 1982.
- [46] M. G. Kreĭn, *Integral equations on a half-line with kernel depending upon the difference of the arguments*. Transl. Amer. Math. Soc. (2) 22 (1962), pp. 163–288. (Translated from Uspekhi Mat. Nauk 13(5) (1958), pp. 3–120.)
- [47] T. Kulczycki, *Properties of Green function of symmetric stable processes*. Probab. Math. Statist. 17(2) (1997) 339–364.
- [48] T. Kulczycki, *Intrinsic ultracontractivity for symmetric stable processes*. Bull. Polish Acad. Sci. Math. 46(3) (1998) 325–334.
- [49] T. Kulczycki, M. Kwaśnicki, J. Małecki, A. Stós, *Spectral Properties of the Cauchy Process on Half-line and Interval*. Proc. London Math. Soc. 30(2) (2010), pp. 353–368.
- [50] T. Kulczycki, B. Siudeja, *Intrinsic ultracontractivity of Feynman-Kac semigroup for relativistic stable processes*. Trans. Amer. Math. Soc. 358(11) (2006) 5025–5057.
- [51] A. Kuznetsov, *On extrema of stable processes*. Ann. Probab. 39(3) (2011), pp. 1027–1060.
- [52] A. Kuznetsov, A. E. Kyprianou, J. C. Pardo, *Meromorphic Lévy processes and their fluctuation identities*. Preprint, 2010, arXiv:1004.4671v2. To appear in Ann. Appl. Probab.
- [53] M. Kwaśnicki, *Spectral analysis of subordinate Brownian motions in half-line. Preliminary version*. Preprint, 2010, arXiv:1006.0524v1.

- [54] M. Kwaśnicki, *Eigenvalues of the fractional Laplace operator in the interval*. Preprint, 2010, arXiv:1012.1133v1.
- [55] M. Kwaśnicki, J. Małecki, M. Ryznar, *Suprema of Lévy processes*. Preprint, 2011, arXiv:1103.0935v1. To appear in Ann. Probab.
- [56] M. Kwaśnicki, J. Małecki, M. Ryznar, *First passage times for subordinate Brownian motions*. Preprint, 2011, arXiv:1110.0401v1.
- [57] A. E. Kyprianou, *Introductory lectures on fluctuations of Lévy processes with applications*. Universitext, Springer-Verlag, Berlin, 2006.
- [58] E. H. Lieb, R. Seiringer, *The Stability of Matter in Quantum Mechanics*. Cambridge University Press, 2010.
- [59] S. A. Molchanov, E. Ostrowski, *Symmetric stable processes as traces of degenerate diffusion processes*. Theor. Prob. Appl. 14(1) (1969), pp. 128–131.
- [60] J. Pitman, M. Yor, *Hitting, occupation and inverse local times of one-dimensional diffusions: martingale and excursion approaches*. Bernoulli 9(1) (2003), pp. 1–24.
- [61] T. Poerschke, G. Stolz, J. Weidmann, *Expansions in generalized eigenfunctions of selfadjoint operators*. Math. Z. 202(3) (1989), pp. 397–408.
- [62] L. C. G. Rogers, *Wiener-Hopf Factorization of Diffusions and Lévy Processes*. Proc. London Math. Soc. 47 (1983), pp. 177–191.
- [63] M. Ryznar, *Estimates of Green Function for Relativistic α -Stable Process*. Potential Anal. 17 (2002), pp. 1–23.
- [64] K. Sato, *Lévy processes and infinitely divisible distributions*. Cambridge Univ. Press, Cambridge, 1999.
- [65] R. Shiraishi, M. Itano, *On the multiplicative products of distributions*. J. Sci. Hiroshima Univ. Ser. A 28 (1964), pp. 223–235.
- [66] R. Schilling, R. Song, Z. Vondraček, *Bernstein Functions: Theory and Applications*. De Gruyter, Studies in Math. 37, Berlin, 2010.
- [67] B. Simon, *Schrödinger semigroups*. Bull. Amer. Math. Soc., New Ser. 7(3) (1982), pp. 447–526.
- [68] R. Song, Z. Vondracek, *Potential theory of special subordinators and subordinate killed stable processes*. J. Theoret. Probab. 19(4) (2006), pp. 817–847.
- [69] R. Song, Z. Vondraček, *On the relationship between subordinate killed and killed subordinate process*. Electronic Communications in Probability 13 (2008), pp. 325–336.
- [70] F. Spitzer, *Some theorems concerning 2-dimensional Brownian motion*. Trans. Amer. Math. Soc. 87 (1958), pp. 187–197.

- [71] V. S. Vladimirov, *Methods of the theory of generalized functions*. Taylor and Francis, New York, 2002.
- [72] N. Wiener, E. Hopf, *Über eine Klasse singulärer Integralgleichungen*. Sitzungber. Akad. Wiss. Berlin (1931), pp. 696–706. (Reprinted in *Norbert Wiener: Collected Works*, Vol. 3, MIT Press, Cambridge, Mass., 1981.)
- [73] A. Zoia, A. Rosso, M. Kardar, *Fractional Laplacian in bounded domains*. Phys. Rev. E 76, 061121 (2007).

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